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Ranjbar-Motlagh, Alireza (IR-SHAR)

An integral type characterization of Lipschitz functions over metric-measure spaces. (English summary)

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If (X, d) is a metric measure space with measure μ, then $f: X \to \mathbb{R}$ is essentially Lipschitz if there exists a constant $L \geq 0$ such that

$$|f(x) - f(y)| \leq L d(x, y) \quad \text{for almost every } x, y \in X.$$

If the above inequality holds for all $x, y \in X$ then f is called Lipschitz.

If the metric measure space is $[0, 1]$ with Euclidean distance and Lebesgue measure, it is known that $f: [0, 1] \to \mathbb{R}$ is essentially Lipschitz if and only if there is $L \geq 0$ such that

$$\int_0^1 \int_0^1 \exp\left(\frac{|f(y) - f(x)|}{L|y - x|} \ln \frac{1}{|y - x|}\right) dy dx < \infty.$$

The present article generalizes this to give an integral characterization of when a measurable function $f: X \to \mathbb{R}$ on a metric measure space is essentially Lipschitz. In this generalization the Euclidean metric and Lebesgue measure are naturally replaced by their metric measure space counterparts.

The result is obtained when the metric measure space is locally strongly Bishop-Gromov regular of some positive dimension k. This is a strong doubling property defined for geodesic spaces, requiring a measure bound as points are moved along geodesics. One of the key tools is a generalized change of variables formula for such spaces.

Gareth Speight

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