# A generalization of L-fuzzy sets and its application in general system theory

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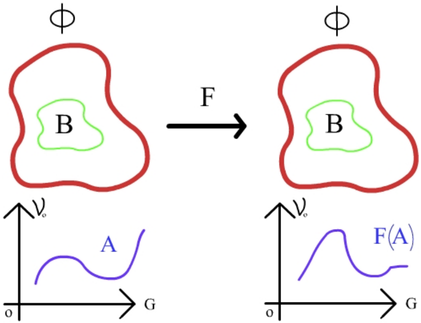
# Basic idea



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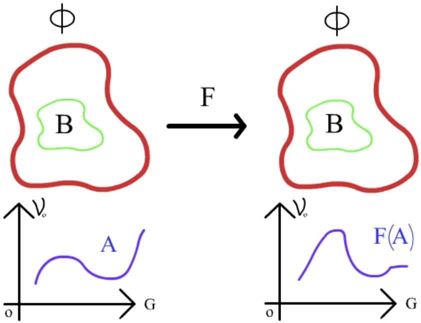


- $\Phi$ is a function space with some nice algebraic and topologic properties (usually completeness conditions).

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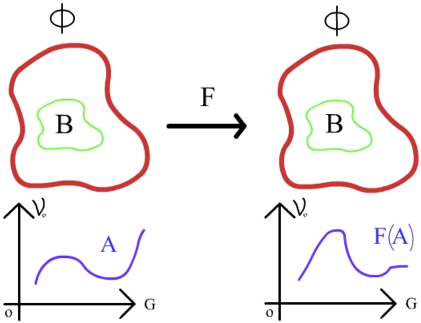
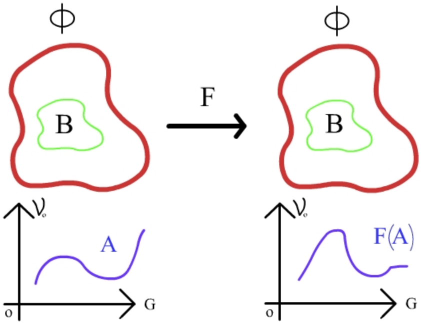



- Specially $\Phi$ is reconstructable, i.e. can be reconstructed properly by a (relatively small) subspace $B$ of its generic objects.

# BASIC IDEA




- Usually $\Phi$ is chosen to be a product space $G \times \mathcal{V}_0$.
  e.g. SPEACH: $G = R$ is the time and $\mathcal{V}_0 = R$ is the space of levels.
  e.g. IMAGE: $G = Z^2$ is the two dimensional space $\mathcal{V}_0 = Z$ is the space of gray-scales.

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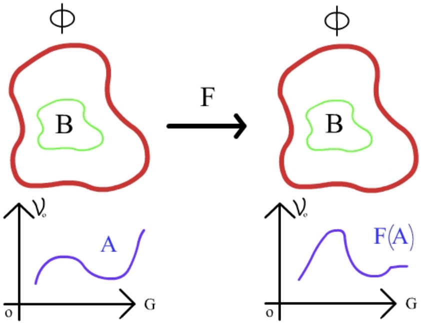


- $F$ is a natural map (i.e compatible with the structure), with nice representation properties.

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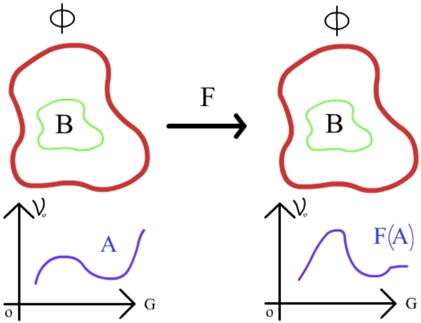


- The whole setup should be in coherence with natural phenomena and be able to simulate input-output behaviour of such systems (this is called I/O system theory).

# A classical example: LSI systems (operators)

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$$F(T_g(A)) = T_g(F(A)),$$

where $T_g$ is the translation-by-$g$ operator, i.e.

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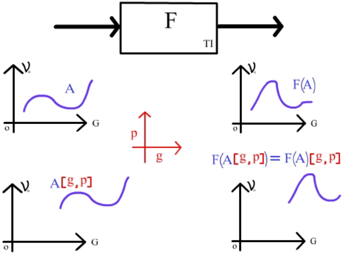
where $T_g$ is the translation-by-$g$ operator, i.e.

$$T_g(A)(t) = A(t - g).$$

- We have the following reconstruction:

$$F(A)(t) = Conv(A, \delta_F) = \int_{-\infty}^{+\infty} \delta_F(\tau) A(t - \tau) d\tau.$$

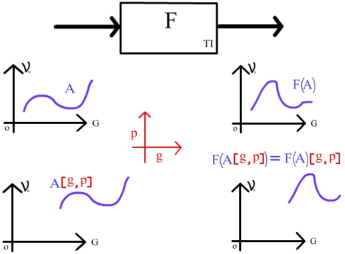
# Translation invariance




- Consider

$$A = \{(t, A(t)) \mid t \in G\}.$$

# Translation invariance



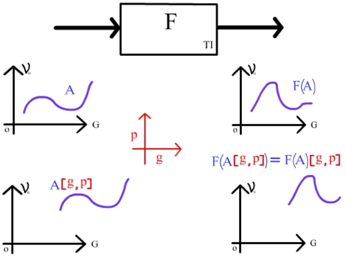

- Then we define the translation operator as follows,

$$A[g,p] = \{(t+g, A(t)*p) \mid t \in G\},$$

which means,

$$A[g,p](t) = A(t-g)*p.$$

# Translation invariance




- An operator $F$ is translation invariant if

$$F(A[g,p]) = F(A)[g,p].$$

Note: For an LSI operator, translation invariance on the range is equivalent to DC-gain 1.

# Outline

# Operators v.s. Comparison

- The set $A = \{(t, A(t)) \mid t \in G\} \subseteq G \times \mathcal{V}_0$ can be considered as a function or a fuzzy set.

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- The difference between the two points of view is in the way we look at the range $\mathcal{V}_0$.
- The functional approach is when we consider algebraic properties and the fuzzy set approach is when we consider the comparative structure (order structure) of $\mathcal{V}_0$.
  e.g. The neutral element for summation is $0$, while the neutral element for the supremum is $-\infty$.

## Minkowski addition and subtraction

- Let $(G, +, -, 0)$ be a group and $\mathcal{V}_0 = (\Omega, \leq, *, \div, 0) \cup \{-\infty, +\infty\}$ be a lattice ordered group with the universal bounds $-\infty$ and $+\infty$ such that,

$$\div(-\infty) = +\infty, \quad \div(+\infty) = -\infty,$$

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- Minkowski addition and subtraction for L-fuzzy sets are defined as follows

$$A \oplus B = \sup_{g} A[g, B(g)] \quad , \quad A \ominus B = \inf_{g} A[g, \div B(g)].$$

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- e.g. Let
  $G =< \ a \mid \ 3a = 0 \ >$,
  $A = \{(0, x_1), (a, x_2), (2a, x_3)\}$,
  $B_1 = \{(0, 0), (a, 0), (2a, -\infty)\}$.
  Then,

  $$A \oplus B_1 = \{(0, \sup(x_1, x_3)), (a, \sup(x_2, x_1)), (2a, \sup(x_3, x_2))\}.$$

# Minkowski erosion as convolution

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where,

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- Minkowski erosion is non-linear and behaves as a convolution operator.

# The kernel

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- e.g. Let
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  $F(A) = \{(0, m), (a, m), (2a, m)\}$, where
  $m = Median(x_1, x_2, x_3)$.
  Then,

$$B(F) = \{\{(0,0), (a,0), (2a, -\infty)\}, \{(0, -\infty), (a, 0), (2a, 0)\}, \{(0,0), (a, -\infty), (2a, 0)\}\}$$

# The reconstruction theorem

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- **Strong Reconstruction Theorem** (Daneshgar 1995)
  Let $F$ be an isotone TI operator. Then

$$F(A) = \sup_{D \in K(F)} Er(A, D);$$

  and if the base of $F$ exists then

$$F(A) = \sup_{B \in B(F)} Er(A, B).$$

# Example: The median filter

- Let $G =< a \mid 3a = 0 >$,
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- Strong reconstruction theorem implies that
  $Median(x_1, x_2, x_3) = F(A)(0)$
  $= \sup(\inf(x_1, x_2), \inf(x_2, x_3), \inf(x_3, x_1))$.

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  Then,

$$B(F) = \{\{(0,0),(a,0),(2a,-\infty)\},\{(0,-\infty),(a,0),(2a,0)\},$$

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  $= \sup(\inf(x_1, x_2), \inf(x_2, x_3), \inf(x_3, x_1))$.
- This is a fuzzification of the Boolean expression
  $x_1x_2 + x_2x_3 + x_3x_1$.

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  - Such limits can be defined in a variety of ways.
  - Convolutions are essentially generalized HOM-functors.
  - Our general setup will cover both fuzzy and functional approach.

# Categorical prerequisites I

- Let $\mathcal{V} = (\mathcal{V}, ., \div, I, a, l, r, c)$ be a symmetric closed monoidal category enriched over itself, with identity $I$, for which $V.- \dashv - \div V$ for any $A \in V$ and the base category $\mathcal{V}_0$ is complete and cocomplete.

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- We also assume that $\mathsf{V} = [I, -]_{\mathcal{V}_0} : \mathcal{V}_0 \longrightarrow$ Set is the base functor and we adapt the multiplicative notation in $\mathcal{V}$.
- On the other hand, let $\mathcal{D}$ be a class of diagram–schemes and assume that $\mathcal{V}_f \subseteq \mathcal{V}_0$ is a small full subcategory such that $\mathcal{V}_0$ is a free $\mathcal{D}$–cocompletion of $\mathcal{V}_f$ in the sense that,
  - The totality of all $\mathcal{D}$–colimits constitute a density presentation for the inclusion $\mathsf{i} : \mathcal{V}_f \hookrightarrow \mathcal{V}_0$.
  - For any $A \in \mathcal{V}_f$, the hom–functor $[A, -]$ preserves all $\mathcal{D}$–colimits.

# Categorical prerequisites II

- we consider a (small) commutative group $(G, +, -)$, with identity $0$ and we focus on the product space $\Phi = \mathcal{V}_0^G$ equipped with the pointwise categorical structure of $\mathcal{V}_0$.

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- It is easy to see that, considering $\Phi$ as a category of functors between the discrete category $G$ and $\mathcal{V}_0$, $\Phi$ inherits the completeness properties of $\mathcal{V}_0$ and that it has a dense $\mathcal{D}$–presentable full subcategory $\Phi_f$.

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- Hereafter, $\Delta_W \in \Phi$ is the constant functor with value $W$.

# Two functors

- Consider the following maps for a fixed $D \in \Phi$,

  $\mathsf{T}_D : \mathrm{obj}(\Phi) \longrightarrow \mathrm{obj}(\Phi),$

  $$\mathsf{T}_D(A)_{(z)} = \prod_{|J|<\infty} \Big(\bigoplus_{j\in J} W_j\Big)^{[A_{(x)}, \oplus_j(\Delta_{W_j} \div D_{(-x+z)})]_\Phi},$$

  $\mathsf{H}_D : \mathrm{obj}(\Phi) \longrightarrow \mathrm{obj}(\Phi),$

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  where $z \in G$ is a fixed coordinate and $\bigoplus$ is a suitable associative bifunctor.
- **Theorem**(Daneshgar& Hashemi 2000)
  Both $\mathsf{T}_D$ and $\mathsf{H}_D$ can be naturally extended to define functors on $\Phi$ for any $D \in \Phi$.

# An important special case

- If $\bigoplus$ is preserved by $(\div)$ (resp. $(.)$) then

$$\mathsf{T}_{D}(A)_{(z)} = \coprod_{x \in G} A_{(x)} . D_{(-x+z)} \overset{\text{def}}{=} \dot{\mathsf{T}}_{D}(A)_{(z)}$$

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- **Theorem**(Daneshgar& Hashemi 2000)
  There exist natural isomorphisms $\tilde{a}, \tilde{l}, \tilde{r}$ and $\tilde{c}$ such that for any $D \in \mathrm{obj}(\Phi)$, $\dot{\Phi} = (\Phi, \dot{\mathsf{T}}_{D}, \dot{\mathsf{H}}_{D}, P^{0,I}, \tilde{a}, \tilde{l}, \tilde{r}, \tilde{c})$ is a symmetric closed monoidal category.

# General reconstruction theorem

- A point $P^{g,V} \in \Phi$ with value $V \in \mathrm{obj}(\mathcal{V}_0)$ at coordinate $g \in G$ is defined (up to isomorphism) as

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- For any $\dot{\Phi}$–functor $\mathsf{F} : \dot{\Phi} \longrightarrow \dot{\Phi}$, $Ker(\mathsf{F})$ is defined to be the category of elements of the Set–functor $[P^{0,I}, F-]_{\Phi}$.
- **Theorem**(Daneshgar& Hashemi 2000)
  Let $\mathsf{F} : \dot{\Phi} \longrightarrow \dot{\Phi}$ be a $\dot{\Phi}$–functor such that $\mathsf{F}$ preserves the internal $\mathrm{Hom}$ of $\dot{\Phi}$, $\dot{\mathsf{H}}_D$, for any $D \in \Phi$; and $Ker(\mathsf{F})$ is a $\mathcal{D}$–type diagram–scheme. Then $\mathsf{F}$ has a representation as a $\mathcal{D}$–colimit of representables as

$$\mathsf{F}(A) \simeq \ \mathrm{Colim}_{(D,d) \in Ker(F)} \ \dot{\mathsf{H}}_D(A).$$

# Example I

- Consider $G = (\mathbf{R}, +)$ and $\mathcal{V} = (\mathbf{R}, +, \leq)$ with two universal bounds $+\infty$ and $-\infty$. Then $\dot{\mathsf{T}}$ is the Minkowski addition and $\dot{\mathsf{H}}$ is the erosion operator.

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- Also, it can be shown that in this case being a $\dot{\Phi}$–functor is equivalent to the definition of a translation invariant operator, where we have

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- Note that in this case the general reconstruction theorem yields the classical reconstruction theorem for TI operators.

# The uniform case

- Now consider the following maps for a fixed $D \in \Phi$,

$$\tilde{\mathsf{T}}_D : \Phi \longrightarrow \mathcal{V},$$

$$\tilde{\mathsf{T}}_D(A) = \prod_{|J|<\infty} \Big(\bigoplus_{j\in J} W_j\Big)^{[A_{(x)}, \oplus_j (\Delta_{W_j} \div D_{(-x+z_j)})]_\Phi},$$

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where $\bigoplus$ is a suitable associative bifunctor.

# A SIMPLE CASE

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If the tensor of $\mathcal{V}$ is preserved by coproducts, then there exist a natural composition law and an identity map such that $[D, B] \stackrel{\text{def}}{=} \ddot{\mathsf{H}}_D(B)$, as the internal $\mathrm{Hom}$, turns $\Phi$ into a $\mathcal{V}$–category.

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- Can you prove a similar theorem for the general case?

# Example II

- Let $G = (\mathbf{R}, +)$ and $\mathcal{V} = (\mathbf{R}^+, ., \geq)$. Then

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- This is the Haar fraction!

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# Some after-thoughts

- In the general setting a closed monoidal category with suitable completeness properties is a suitable model for the valuation space that generalizes both functional and fuzzy approaches.
- The fuzzy set approach introduces applications based on Boolean functions that give rise to non-linear operators.
- Can one define a limit process that simulates integration in the general setting?
- Can one develop transformation techniques that facilitate a design process based on some wanted behaviour of operators. (This should be definitely related to some new understandings and dimensions in system design.)

# The End.

Thank You!