Spectral extremality, tensor-like constructions and commutativity in graphs

Amir Daneshgar
Sharif University of Technology

http://www.sharif.ir/~daneshgar

26 December 2016 (6 Dey 1395)

daneshgar@sharif.ir

Outline

Primes and Zeta Functions: definitions

- Consider a geometric space made of primes and their amalgams.
- e.g. $\mathbb{Q}$, number fields, function fields, Riemannian manifolds, graphs, ...
- The connectivity of the space is naturally related to number of primes and how they are mixed together.
- Connectivity is a fundamental concept that can be studied and measured in many different ways.
- A zeta (in general L) function is a mathematical concept that is supposed to present and reflect all these aspects in a reasonable way!

Geometric spaces and zeta functions

A zeta function assigned to a geometric space is a generating function with many nice and informative representations related to the space.

This generating function

- contains a categorized dictionary of primes and their rate of appearance
- can be represented in many different informative ways
- is related to the fundamental group of the space
- is related to fundamental linear dynamics on the space defined using natural cohomologies.
- has nice functional properties

Primes and Zeta Functions: cohomology

Rational numbers

The Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \ prime} (1 - p^{-s})^{-1}$ is related to Hecke operators but possibility for relation to a natural diffusion is not fully understood yet.

Graphs

The Ihara zeta function $\zeta(u) = \prod_{[P] \ prime} (1 - u^{\ell([P])})^{-1}$ is related to the adjacency operator and this relation is fully understood.

Apply $u := q^{-s}$ to compare!

Pseudomathematics

$$\log \zeta(u) = -\sum_{[P]} \log(1 - u^{\ell([P])}) = \cdots = \sum_{m \geq 1} \frac{N_m}{m} u^m$$

$$= \sum_{m \geq 1} \frac{u^m}{m} tr(B^m) = tr\left(\sum_{m \geq 1} \frac{u^m}{m} B^m \right) = tr\left(\log(I - uB)^{-1}\right)$$

$$= \log\left((\det(I - uB))^{-1} \right)$$

D. Hilbert: Does there exists such a natural operator B for Riemann's zeta function?

Riemann Hypothesis and Highly Connected Objects

- Poles of this zeta function are essentially the eigenvalues of B!
- B is usually (cohomologically) related to a natural dynamics on the object.
- Lesser spread of spectrum for B gives rise to faster dynamics/diffusion, and hence, more connectivity!
- There are mysterious connections between zeros of Riemann's zeta function for $\mathbb{Q}$ and spectra of random matrices!
- The case of function fields have been extensively studied as a feasible case (1949-1974: Weil conjectures proved by Deligne).
- The case of graphs is equally important, accessible, and interesting. The chance of applying new combinatorial techniques is already verified!

A Royal Road to Mathematics

Euclid:

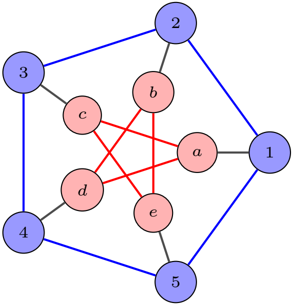
There is no Royal Road to geometry.

Graph theory zoo

Graph theory is a Royal Road to the heart of modern mathematics that is free to be used by any curious scholar!

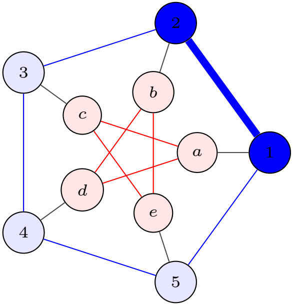
The road passes through Computer Science land of Oz!

Graphs and Matrices



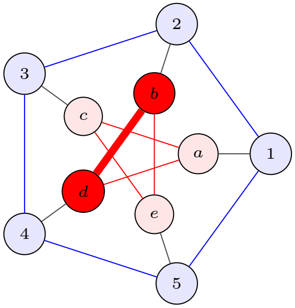
$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Graphs and Matrices



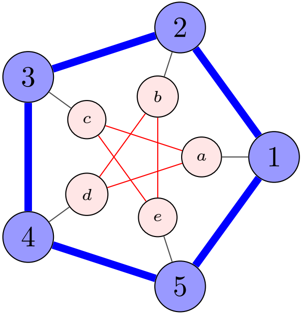
$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Graphs and Matrices



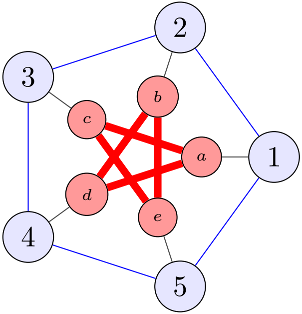
$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Graphs and Matrices



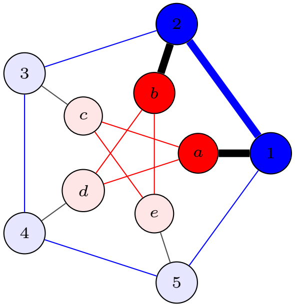
$$\begin{bmatrix} . & 1 & . & . & 1 & 1 & . & . & . & . \\ 1 & . & 1 & . & . & . & 1 & . & . & . \\ . & 1 & . & 1 & . & . & . & 1 & . & . \\ . & . & 1 & . & 1 & . & . & . & 1 & . \\ 1 & . & . & 1 & . & . & . & . & . & 1 \\ 1 & . & . & . & . & . & . & 1 & 1 & . \\ . & 1 & . & . & . & . & . & . & 1 & 1 \\ . & . & 1 & . & . & 1 & . & . & . & 1 \\ . & . & . & 1 & . & 1 & 1 & . & . & . \\ . & . & . & . & 1 & . & 1 & 1 & . & . \end{bmatrix}$$

Graphs and Matrices



$$\begin{bmatrix} . & 1 & . & . & 1 & 1 & . & . & . & . \\ 1 & . & 1 & . & . & . & 1 & . & . & . \\ . & 1 & . & 1 & . & . & . & 1 & . & . \\ . & . & 1 & . & 1 & . & . & . & 1 & . \\ 1 & . & . & 1 & . & . & . & . & . & 1 \\ 1 & . & . & . & . & . & . & 1 & 1 & . \\ . & 1 & . & . & . & . & . & . & 1 & 1 \\ . & . & 1 & . & . & 1 & . & . & . & 1 \\ . & . & . & 1 & . & 1 & 1 & . & . & . \\ . & . & . & . & 1 & . & 1 & 1 & . & . \end{bmatrix}$$

Graphs and Matrices



$$
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}
$$

Graphs and Their Laplacians

- Let $G = (V, E)$ be a finite simple graph with the adjacency matrix A.
- Let D be the diagonal matrix of degrees.
- The Laplacian of G is defined to be the matrix $\Delta \stackrel{\text{def}}{=} D - A$.
- Note that for d-regular graphs the Laplacian is $dI - A$.
- Laplacian is the natural operator related to energy.
- There exists a natural matrix ∇ representing differentiation such that the Laplacian can be represented as $\nabla^t \nabla$. (can you find it?)
- Laplacian is related to natural diffusions on G.
- Ihara: $\zeta_G(u)^{-1} = (1 - u^2)^{|E|-|V|} \det(I - Au + (D - I)u^2)$.

Ramanujan Graphs

Riemann Hypothesis (RH) for $\mathbb{Q}$

If $\zeta(s) = 0$ and $0 < Re(s) < 1$ then $Re(s) = 1/2$.

Riemann Hypothesis (RH) for graphs (Ihara zeta func.)

If $\zeta(q^{-s})^{-1} = 0$ and $0 < Re(s) < 1$ then $Re(s) = 1/2$.

This is equivalent to the following:

Ramanujan graphs

A $(q+1)$-regular graph with adjacency matrix A satisfies RH iff it is Ramanujan, i.e. if

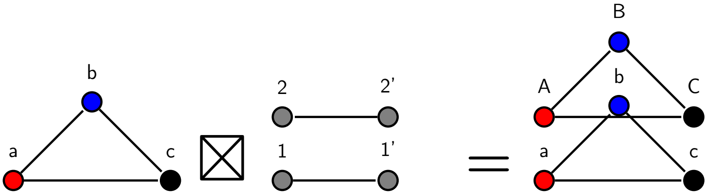
$$\mu \overset{\text{def}}{=} \max\{|\lambda| \mid \lambda \in Spec(A) \;\&\; |\lambda| \neq q+1\}$$

then $\mu \leq 2\sqrt{q}$.

Expanders

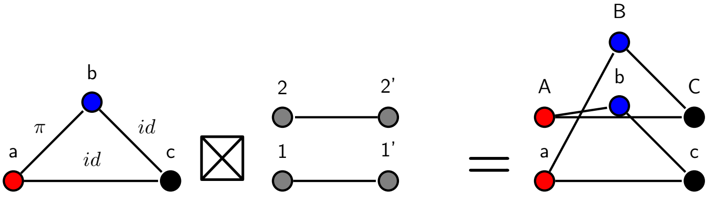
- Eigenvalues are extremal solutions for a variational problem on the normalized energy $\frac{\langle \Delta f, f \rangle}{\langle f, f \rangle} = \frac{\|\nabla f\|_2^2}{\|f\|_2^2}$.
- Isoperimetric numbers are extremal solutions for a variational problem on the normalized flow $\frac{\|\nabla f\|_1}{\|f\|_1}$.
- Comparison between these parameters are known as isoperimetric (in particular for the first parameter as Cheeger-Maz'ya) inequalities.
- Noting that eigenvalues can be computed in polynomial time, these inequalities can be interpreted as approximation estimates for the NP-hard isoperimetry problem.
- Expansion is the simplest form of isoperimetry used to approximate high connectivity.

Ordinary Tensors for Adjacency Matrices



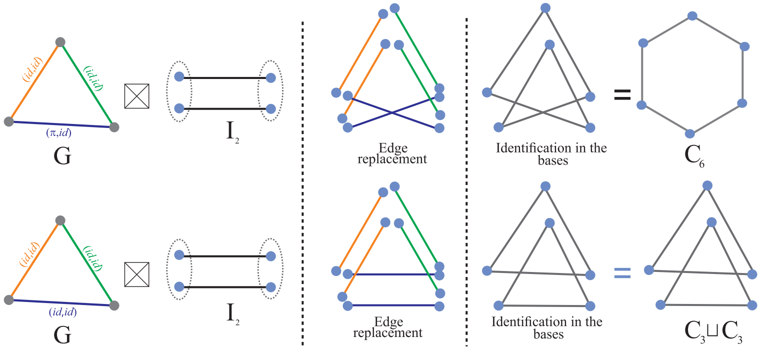
$$
\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \end{array} \begin{array}{c} a \\ b \\ c \end{array}
\boxtimes
\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array}
=
\begin{array}{c} \begin{array}{cccccc} a & A & b & B & c & C \end{array} \\ \left(\begin{array}{cc|cc|cc} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right) \end{array} \begin{array}{c} a \\ A \\ b \\ B \\ c \\ C \end{array}
$$

Ordinary Tensors for Adjacency Matrices



$$\overset{\begin{matrix} a & b & c \end{matrix}}{\begin{pmatrix} 0 & {\color{red}\pi} & 1 \\ {\color{red}\pi} & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}} \begin{matrix} a \\ b \\ c \end{matrix} \boxtimes \overset{\begin{matrix} 1 & 2 \end{matrix}}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \begin{matrix} 1 \\ 2 \end{matrix} = \overset{\begin{matrix} a & A & b & B & c & C \end{matrix}}{\left(\begin{array}{cc|cc|cc} 0 & 0 & {\color{red}\mathbf{0}} & {\color{red}\mathbf{1}} & 1 & 0 \\ 0 & 0 & {\color{red}\mathbf{1}} & {\color{red}\mathbf{0}} & 0 & 1 \\ \hline {\color{red}\mathbf{0}} & {\color{red}\mathbf{1}} & 0 & 0 & 1 & 0 \\ {\color{red}\mathbf{1}} & {\color{red}\mathbf{0}} & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right)} \begin{matrix} a \\ A \\ b \\ B \\ c \\ C \end{matrix}$$

Lifts and Randomness: random 2-lifts



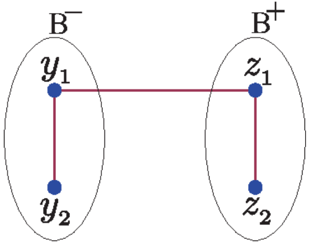
Lifts and Randomness: random 2-lifts

A. W. Marcus, D. A. Spielman, N. Srivastava, *Ann. Math.* (2015)

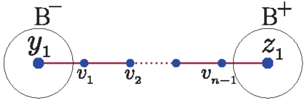
The exists regular Ramanujan graphs of arbitrary degree within the iterated random 2-lifts of complete graphs.

The proof is based on the fundamental technique of interlacing families of polynomials which is also used by the same authors to prove Kadison-Singer Problem.

Cylindrical Construction: examples

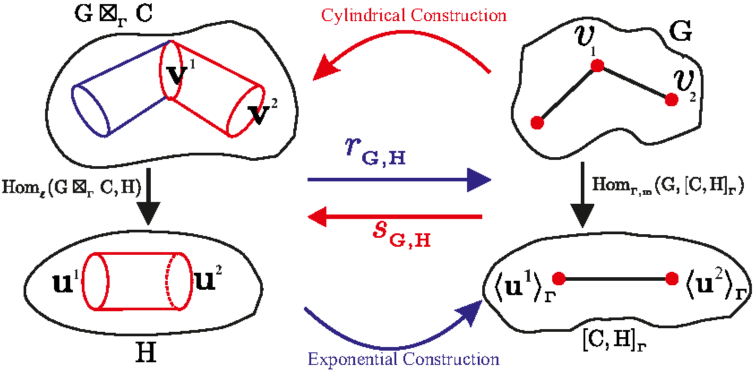


The π-cylinder,



Path cylinder P_n

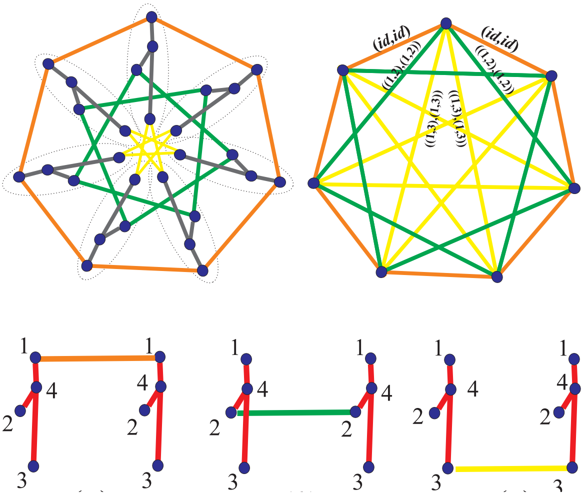
Schematic Duality Diagram



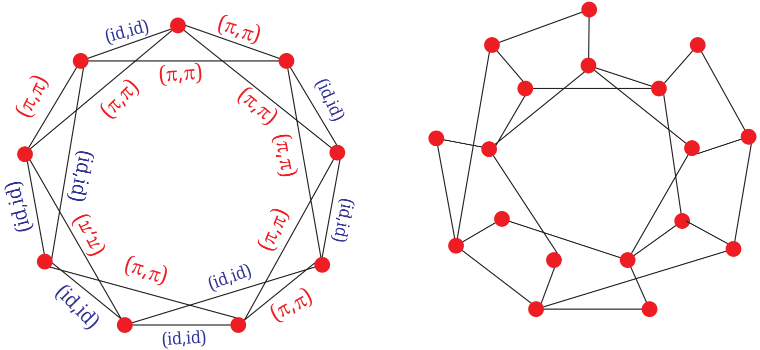
Tree-cylinders (the Petersen graph)

The Petersen graph Show

Tree-cylinders (the Coxeter graph)

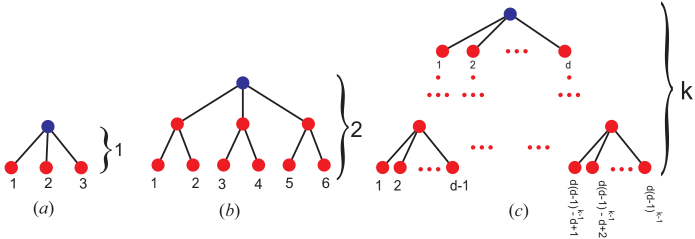


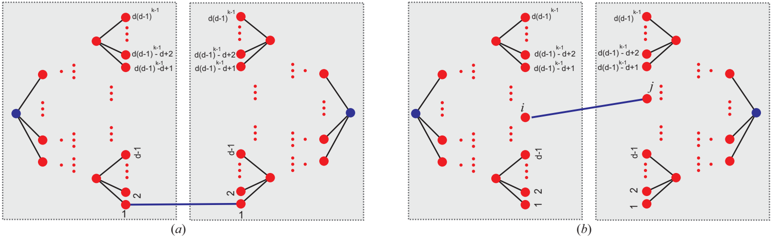
Cylindrical Construction: random π-lifts



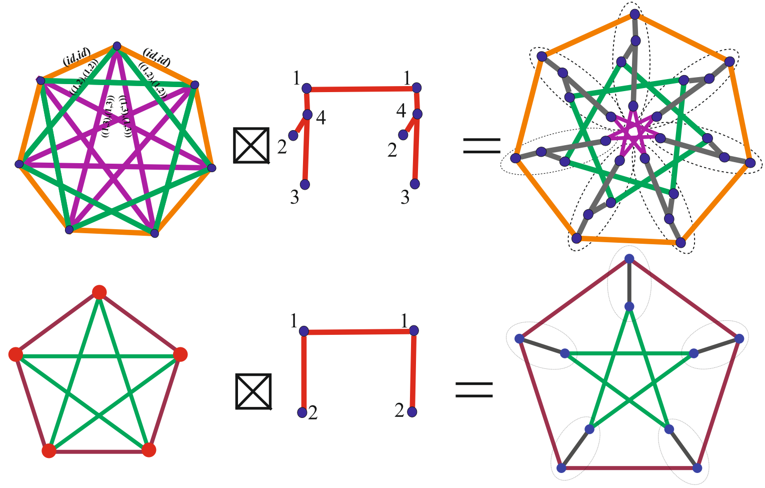
(a) (b)

Tree Cylinders: how they help?





Some Lifts of Complete Graphs: definition



Bilateral Symmetry, Commutative Decompositions and Spectrum

Let $\mathcal{H}$ be a symmetric cylinder with no internal vertices (e.g. a tree-cylinder), then

A spectral result

$$\phi(\mathcal{G}\boxtimes\mathcal{H}, x) = \prod_{j=1}^{n} \phi\left(B + \sum_{i=0}^{t-1} \theta_i^j E_i^{bb'}, x\right),$$

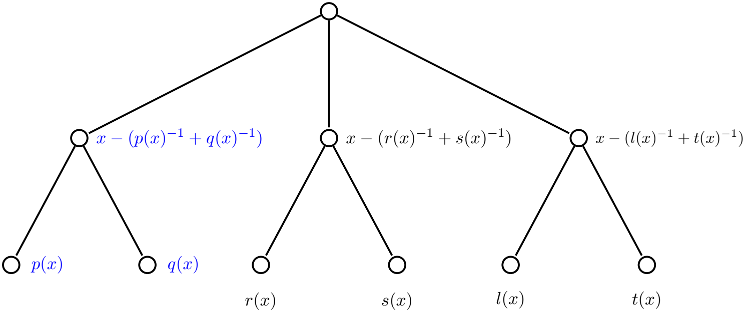
in which, B is the base of the cylinder, ϕ is the characteristic polynomial and sum is a term depending on the partition.

Summary!

The spectrum of such a construction is a perturbation of the spectrum of the base depending on the construction and the twists.

Eigenvalue Mixing

do the same!



This is essentially how the determinant of a perturbation of a tree can be computed in most important cases!

T-graphs

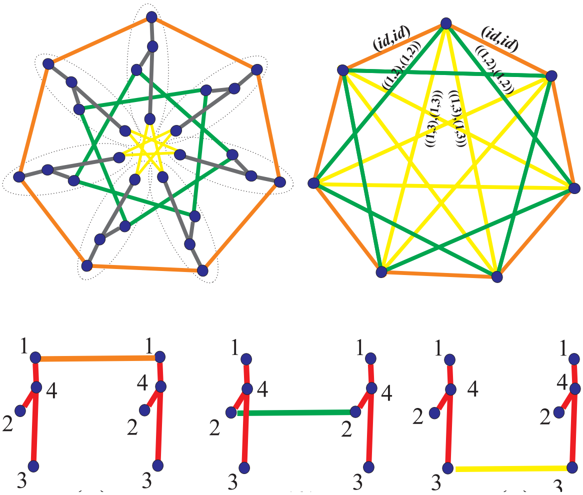
History

Tree cylinders of M. Madani + A. Taherkhani $\Rightarrow$ T-graphs!

Definition

A T-graph is a cylindrical construct that can be described as replacing each vertex of a complete graph by a complete tree and join the leaves in a special predefined order called group labeling of trees.

Examples of T-graphs (the Coxeter graph)



A 3-regular Ramanujan graph of order 130

Setup

Take the 3-regular tree of hight 2 with 6 leaves as the base of the tree-cylinders and choose the complete graph on 13 vertices as the base-graph of the construction.

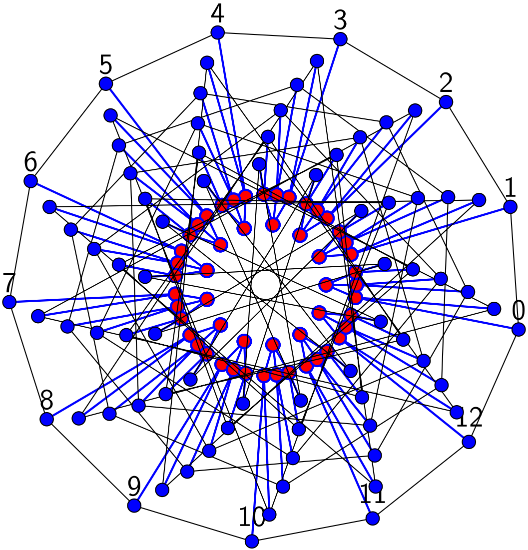
Using group-labeling this gives rise to a 3-regular Ramanujan graph of order 130 with the following characteristic polynomial,

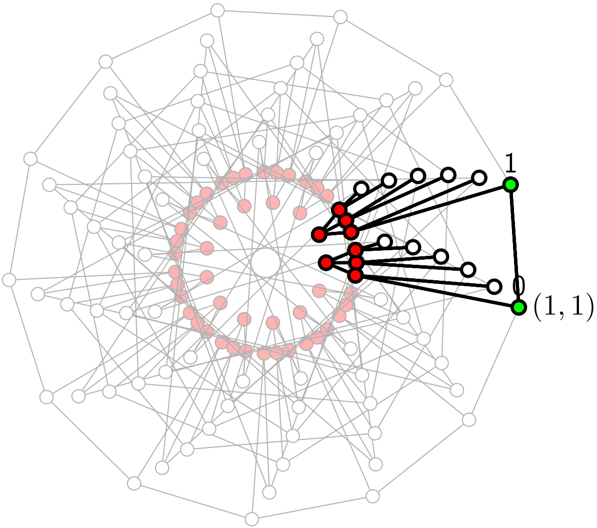
$$\phi(\mathcal{K}_{13} \boxtimes \mathcal{H}^{\bullet}, x) = (x-3)(x-1)(x+2)(x-2)^3(x^2-2x-2)^2$$

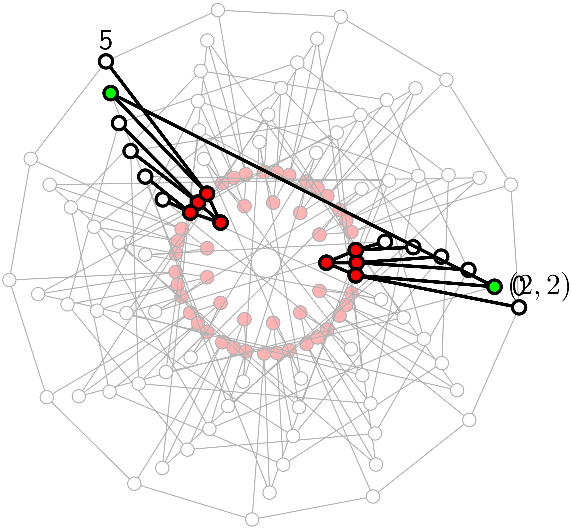
$$\times(x^{10}+x^9-14x^8-12x^7+65x^6+45x^5-115x^4-55x^3+69x^2+12x-10)^{12}.$$

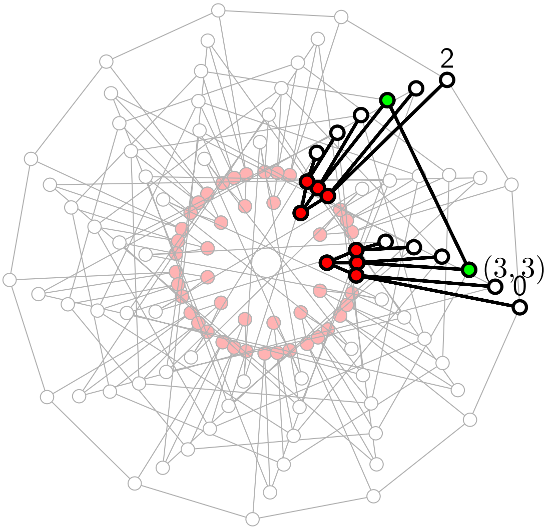
Roots:

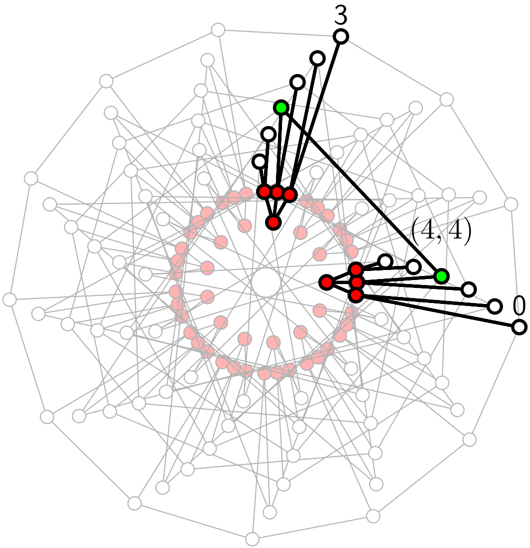
$[-2.635(12), -2.197(12), -2.000, -1.603(12), -1.135(12), -0.732(2), -0.485(12), 0.396(12), 0.670(12), 1, 1.424(12), 2(3), 2.08(12), 2.485(12), 2.732(2), 3.000]$

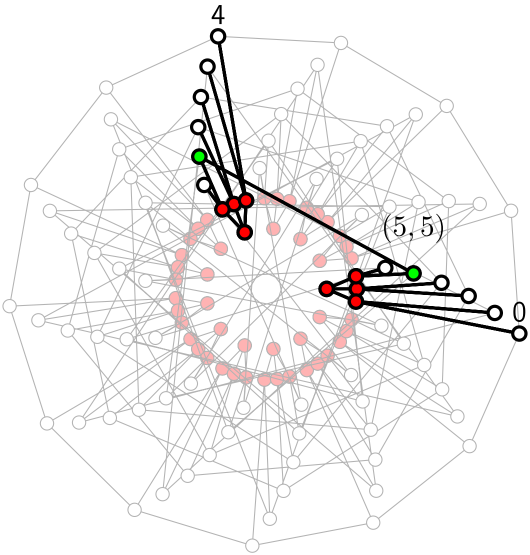

4
3
5
2
6
1
7
0
8
12
9
11
10

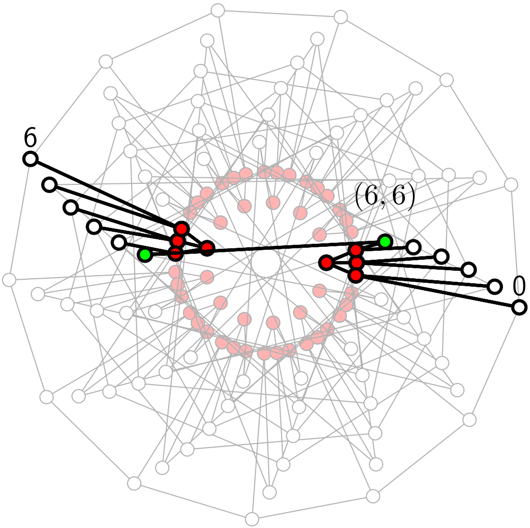

1
0
(1, 1)


5
(2, 2)


2
(3, 3)
0


3
(4, 4)
0


4
(5, 5)
0


6
(6, 6)
0

Some Questions to Answer!

- Analyze the spectra of T-graphs.
- Analyze the roots of the polynomial which is the result of an eigenvalue mixing on a tree.
- Prove that there exists nice 3-regular Ramanujan graphs within the iterated π-lifts of complete graphs. (note: this is supported by our experimental results[1].)

[1] Courtesy of Kasra Alishahi

Thank you!

Comments and Criticisms are Welcomed

daneshgar@sharif.ir