Tropical Algebras, Idempotent Analysis and System Theory

A Story of Continuous Mathematics in The Limit

Amir Daneshgar

Department of Mathematical Sciences
Sharif University of Technology

http://www.sina.sharif.ir/∼daneshgar
daneshgar@sharif.ir

Bahman 1393 (February, 2015)

Outline

1. Prologue: Good Mathematics Over Semirings!

OUTLINE

Outline

1. Prologue: Good Mathematics Over Semirings!
2. Semirings: Rings and Dioids
3. A Couple of Examples

Outline

Outline

1. Prologue: Good Mathematics Over Semirings!
2. Semirings: Rings and Dioids
3. A Couple of Examples
4. Tropical Algebra/Geometry
5. Idempotent Analysis

Outline

[D. Hilbert, 1900]
“Mathematical science is in my opinion an indivisible whole, an organism whose vitality is conditioned upon the connection of its parts.”

Prologue:
Good Mathematics
Over
Semirings!

Good Mathematics Over Semirings!

[J. von Neumann, *The Mathematician* 1947]

"As a mathematical discipline travels far from its empirical source, or still more, if it is a second and third generation only indirectly inspired from ideas coming from 'reality', it is beset with very grave dangers. ... But there is a grave danger that the subject will develop along the line of least resistance, that the stream, so far from its source, will separate into a multitude of insignificant branches, and that the discipline will become a disorganized mass of details and complexities."

Good Mathematics Over Semirings!

[T. Tao, *What is Good Mathematics* 2007]
"However, there is the remarkable phenomenon (ref. Wigner *unreasonable effectiveness of mathematics*) that good mathematics in one of the above senses tends to beget more good mathematics in many of the other senses as well, leading to the tentative conjecture that perhaps there is, after all, a universal notion of good quality mathematics, and all the specific metrics listed above represent different routes to uncover new mathematics, or different stages or aspects of the evolution of a mathematical story."

Good Mathematics Over Semirings!

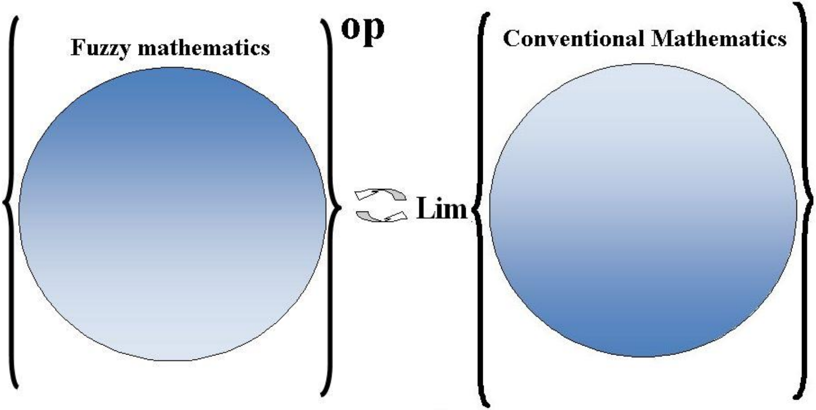
GENERAL REFERENCES FOR THE ORIGINS:

- Barvinok Alexander I.; *Combinatorial Optimization and Computations in the Ring of Polynomials*, DIMACS Technical Report, 93-13, 1993.
- Bergstra J. A. and J. W. Klop; *The algebra of recursively defined processes and the algebra of regular processes*, in J. Paredaens (ed.): Automata, Languages and Programming, Lecture Notes in Computer Science, 172, Springer- Verlag, Berlin, 1984.
- Cuninghame-Green Raymond A.; *Minimax Algebra*, Lecture Notes in Economics and Mathematical Systems, 166, Springer-Verlag, Berlin, 1979.
- Dedekind Richard; *Über die Theorie der ganzen algebraiscen Zahlen*, Supplement XI to P. G. Lejeune Dirichlet: Vorlesung über Zahlentheorie 4 Aufl., Druck und Verlag, Braunschweig, 1894.
- Eilenberg Samuel; *Automata, Languages, and Machines*, vol A, Academic Press, New York, 1974.
- Golan Jonathan S.; *Semirings and Their Applications*, Kluwer, Dordrecht, 1999.
- Gondran M. and M. Minoux; *Graphes et algorithmes*, Editions Eyrolles, Paris, 1979, 1988.
- Gondran M. and M. Minoux; *Graphs, Dioids and Semirings: New Models and Algorithms*, Springer Science & Business Media LLC, 2008.
- Maslov Victor P.; *New superposition principle for optimization problems*, In: Seminaire sur les Equations aux Dérivées Partielles 1985/86, Centre Math. De ÍEcole Polytechnique, Palaiseau, 1986, exposé 24.
- Mikhalkin Grigory; *Real algebraic curves, the moment map and amoebas*, Ann. Math. (2) 151 (2000), 309-326.
- Salomaa Arto and Matti Soittola; *Automata-theoretic Aspects of Formal Power Series*, Springer-Verlag, Berlin, 1978.
- Kuich Werner and Arto Salomaa; *Semirings, Automata, Languages*, Springer London, 20011.
- Simon Imre ; *Recognizable sets with multiplicities in the tropical semiring*, in M. P. Chytil et al. (eds.): Mathematical Foundations for Computer Science, 1988, Lecture Notes in Computer Science, 324, Springer-Verlag, Berlin, 1988.
- Vandiver H. S.; *Note on a simple type of algebra in which cancellation law of addition does not hold*, Bull. Amer. Math. Soc. 40 (1934), 914-920.
- Viro Oleg; *Dequantization of real algebraic geometry on a logarithmic paper*, in Proceedings of the 3rd European Congress of Mathematicians, Progress in Mathematics, 201, Birkhaüser Verlag, Basel, 2001.
- Wechler Wolfgang; *R-fuzzy computation*, J. Math. Anal. Appl. 115 (1986), 225-232.

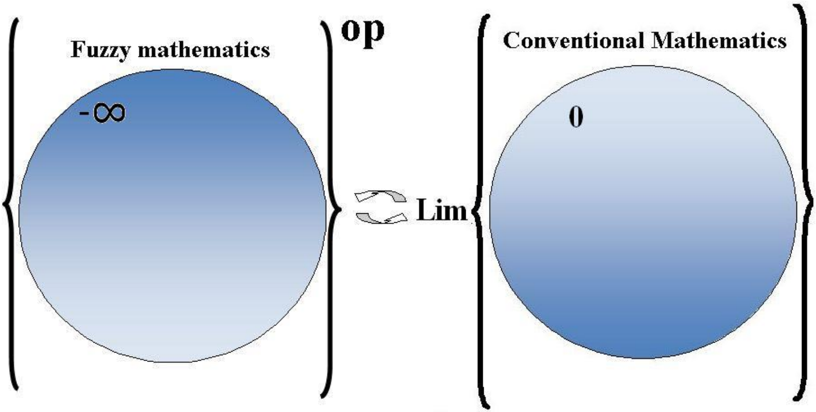
Good Mathematics Over Semirings!

Fuzzy set theory	Lotfi AskarZadeh	1965,
	Joseph Goguen	1968/9
	Wolfgang Wechler	1983,
	Jonathan S, Golan	1999
Theory of Computation	Samuel Eilenberg	1974,
	Arto Salomaa	1978,
	Jan A. Bergstra, Jan W. Klop	1984,
Discrete Optimization	Raymond A. Cuninghame-Green	1979,
	Michel Gondran, Michel Minoux	1984,
	Alexander I. Barvinok	1993
Idempotent Analysis	Victor P. Maslov, Grigory L. Litvinov	1986,
Tropical Algebra/Geometry	Imre Simon	1988,
	Grigory Mikhalkin, Oleg Viro	2000.

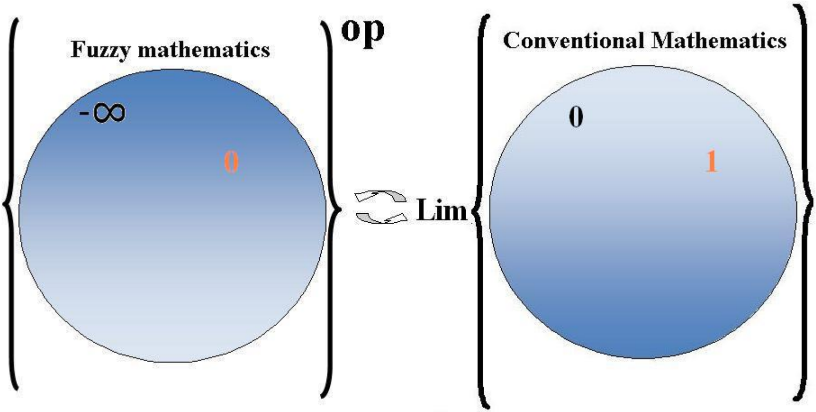
The mysterious duality!



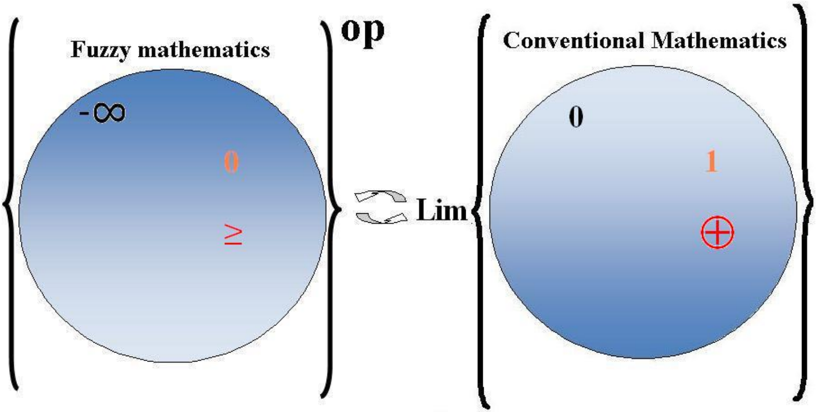
The mysterious duality!



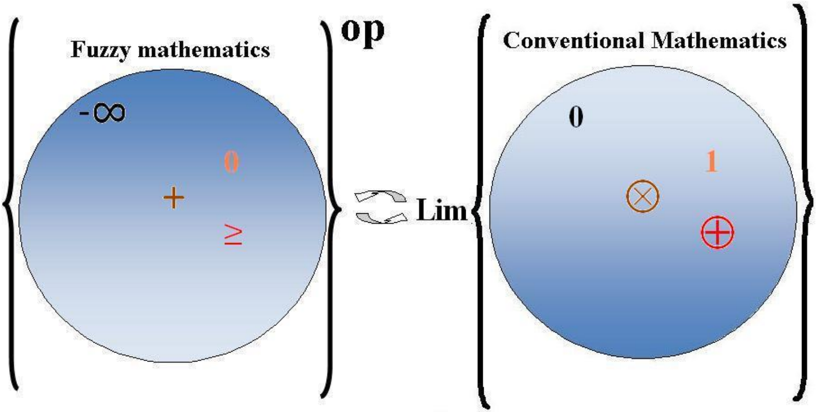
The mysterious duality!



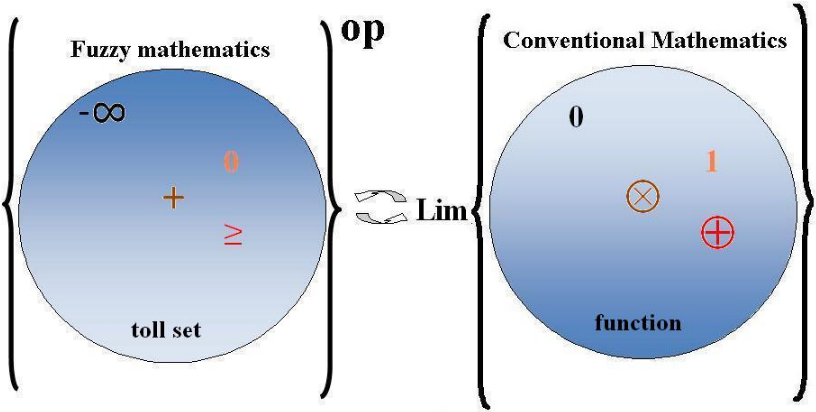
The mysterious duality!



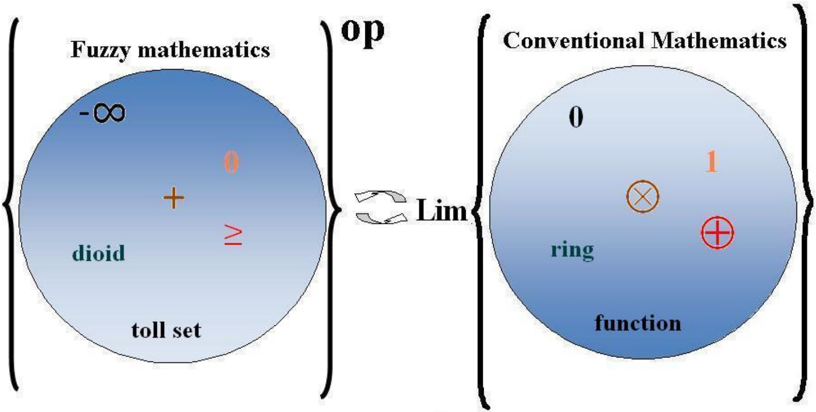
The mysterious duality!



The mysterious duality!



The mysterious duality!



Good Mathematics Over Semirings!

- Is there any difference between *mathematics over fields* and *mathematics over semirings*?

Let's talk about this!

Semirings,

Rings and Dioids

Definition of a semiring

- A **semiring** $(S, \oplus, \otimes, \mathbf{1}, \mathbf{0})$ is an algebraic structure which satisfies the following properties:
  1. $\oplus$ and $\otimes$ are associative binary operations on S,
  2. $(S, \oplus, \mathbf{0})$ is a commutative monoid with identity $\mathbf{0}$,
  3. $(S, \otimes, \mathbf{1})$ is a monoid with identity $\mathbf{1}$.
  4. for all $a, b, c \in S$ we have

     $$a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c) \text{ and } (b \oplus c) \otimes a = (b \otimes a) \oplus (c \otimes a).$$

  5. for all $a \in S$ we have $a \otimes \mathbf{0} = \mathbf{0} \otimes a = \mathbf{0}$.

Subclasses of semirings

The class of semirings can be naturally subdivided into two disjoint sub-classes depending on whether the operation $\oplus$ satisfies one of the following two properties:

1. The operation $\oplus$ endows the set D with a group structure;
2. The operation $\oplus$ endows the set D with a canonically ordered monoid structure.

note

(1) and (2) cannot be satisfied simultaneously. In case (1), we are led to the well-known Ring structure, and in case (2) we are lead to the Dioid structure.

Definition of a dioid

- A **Dioid** is a quintuple $(D, \vee, \otimes, \mathbf{1}, \mathbf{0})$ that satisfies the following conditions:
  1. $(D, \vee)$ is a commutative monoid with neutral element $\mathbf{0}$.
  2. $(D, \otimes)$ is a monoid with neutral element $\mathbf{1}$.
  3. for all $d \in D$ we have $d \otimes \mathbf{0} = \mathbf{0} \otimes d = \mathbf{0}$.
  4. $\otimes$ is (right and left) distributive with respect to $\vee$.
  5. $\vee$ induces an order define as:

$$a \leq b \Leftrightarrow \exists\, c, \;\; b = a \vee c.$$

note

A dioid is a semiring $(D, \vee, \otimes, \mathbf{1}, \mathbf{0})$ in which the operation $\vee$ induces a natural order structure.

Idempotent dioids

note

In general in a dioid $(D, \vee, \otimes, \mathbf{1}, \mathbf{0})$ the canonical order is compatible with the operations $\vee$ and $\otimes$.

A dioid is **idempotent** iff $\forall\, a \in D,\ a \vee a = a$.
Note that this implies the following fact,

$$a \leq b \Leftrightarrow \exists\, c,\ \ b = a \vee c \Leftrightarrow a = a \vee b.$$

note

Sometimes this condition is added to the definition of a dioid as an axiom.

A Couple of

Motivating Examples

Examples of dioids

Ideals of a commutative ring	R. Dedekind 1894
	H. S. Vandiver 1934
$(\mathbb{N}, +, \times)$	A dioid which is not idempotent
$(\mathbb{N} \cup \{+\infty\}, \min, +)$	Tropical semiring
$(\mathbb{Z}, +, \times)$	A ring but not a dioid.
$(\mathbb{N}, lcm, gcd)$	A doubly idempotent dioid.
$\mathbb{R}_{\min} \stackrel{\text{def}}{=} (\mathbb{R} \cup \{+\infty\}, \min, +)$	Min-plus (optimization) algebra
$\mathbb{R}_{\max} \stackrel{\text{def}}{=} (\mathbb{R} \cup \{-\infty\}, \max, +)$	Max-plus algebra.
$\overline{\mathbb{R}} \stackrel{\text{def}}{=} (\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}, \max, \min)$	An idempotent dioid.
$(Reg(\Sigma), \cup, concat)$	A dioid (used in Computer Science).
$(2^{\mathbb{R}^2}, \cup, +)$	A dioid (used in Integral Geometry)

$A + B \stackrel{\text{def}}{=} \{y + z \ : \ y \in A, \ \& z \in B\}.$

Has mathemtics gone wild for a century?!

Polynomials in $\mathbb{R}_{min}[x,y] \overset{\text{def}}{=} (\mathbb{R} \cup \{+\infty\}, \oplus \overset{\text{def}}{=} \min, \otimes \overset{\text{def}}{=} +)[x,y]$

A monomial

$ax^iy^j \overset{\text{def}}{=} a \otimes x \ldots \otimes x \otimes y \ldots \otimes y = a + ix + jy.$

A polynomial

$ax^2 + bxy + e = \min(a + 2x, b + x + y, e).$

Neutrals

$\infty + x = 0x = x.$

Fact:

The tropical polynomials in n variables $x_1, \ldots, x_n$ are precisely the piecewise-linear concave functions on $\mathbb{R}^n$ with integer coefficients.

Polynomials in $\mathbb{R}_{min}[x,y] \stackrel{\text{def}}{=} (\mathbb{R}\cup\{+\infty\}, \oplus \stackrel{\text{def}}{=} \min, \otimes \stackrel{\text{def}}{=} +)[x,y]$

Freshman's dream!

$$3\min(x,y) = (x\oplus y)^3 = x^3 \oplus y^3 = \min(3x,3y).$$

More surprises!

$$(x+1)^2 = x^2 + 17x + 2 = x^2 + 1x + 2.$$

Fundamental theorem of algebra

- Every tropical polynomial function can be written uniquely as a tropical product of tropical linear functions.
- Unique factorization of tropical polynomials holds in one variable, but it no longer holds in two or more variables.

Polynomials in $\mathbb{R}_{min}[x, y] \stackrel{\text{def}}{=} (\mathbb{R} \cup \{+\infty\}, \oplus \stackrel{\text{def}}{=} \min, \otimes \stackrel{\text{def}}{=} +)[x, y]$

Cubic polynomials in one variable

Let $b - a \leq c - b \leq d - c$, then

$$ax^3 + bx^2 + cx + d = a(x + (b - a))(x + (c - b))(x + (d - c)).$$

Matrix multiplication in $\mathbb{R}_{max}$

Example

For a matrix $A = \begin{pmatrix} 3 & 7 \\ 2 & 4 \end{pmatrix}$ in $Mat_{n \times n}(\mathbb{R}_{max})$ we have

$$A^2 = \begin{pmatrix} max(3+3, 7+2) & max(3+7, 7+4) \\ max(2+3, 4+2) & max(2+7, 4+4) \end{pmatrix} = \begin{pmatrix} 9 & 11 \\ 6 & 9 \end{pmatrix}.$$

In linear algebra we have

$$A^2 = \begin{pmatrix} +(3 \times 3, 7 \times 2) & +(3 \times 7, 7 \times 4) \\ +(2 \times 3, 4 \times 2) & +(2 \times 7, 4 \times 4) \end{pmatrix} = \begin{pmatrix} 23 & 49 \\ 14 & 30 \end{pmatrix}.$$

Matrix multiplication in $\mathbb{R}_{min}$

The shortest path problem

Let $A \in Mat_{n\times n}(S)$ be the adjacency matrix of the weighted directed graph G on the vertex set $\{v_1, v_2, \ldots, v_n\}$. Then the shortest path problem is equal to find the matrix $D = [d_{ij}]$ such that d_{ij} is the minimum of the weights of all paths starting from v_i and ending at v_j. Note that in this setting we have,

$D = A^* = I + A + A^2 + \cdots + A^k + \cdots = A^{n-1}$.

Metrics

A is said to be a metric if $A^2 = A$.

(Note that this is the triangle inequality!)

Medians in $\overline{\mathbb{R}} \stackrel{\text{def}}{=} (\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}, max, min)$

The median filter on $\overline{\mathbb{R}}^{\mathbb{Z}_3}$

Let

$$\overline{\mathbb{R}}^{\mathbb{Z}_3} \ni A = \{(0, x_1), (1, x_2), (2, x_3)\},$$

and let M be the median filter acting as

$$M(A) = \{(0, m), (1, m), (2, m)\},$$

where $m = Median(x_1, x_2, x_3)$.
Note that $m = \max(\min(x_1, x_2), \min(x_2, x_3), \min(x_3, x_1))$, which is a fuzzification of the Boolean median expression $x_1x_2 + x_2x_3 + x_3x_1$.

Tropical Algebra/Geometry

Tropical Geometry

GENERAL REFERENCES FOR FURTHER READING:

- Itenberg Illia, Grigory Mikhalkin, Eugenii Shustin; *Tropical algebraic geometry*, 2nd ed. Birkhäuser Verlag AG, 2009.
- Gathmann Andreas and Hannah Markwig; *Kontsevich's formula and the WDVV equations in tropical geometry*, Adv. Math. 217 (2008), no. 2, 537-560.
- Gelfand I. M., M. M. Kapranov, and A. V. Zelevinsky; *Discriminants, resultants, and multidimensional determinants*, Mathematics: Theory & Applications. Birkhäuser Boston Inc., Boston, MA, 1994.
- Izhakian Z. and L. Rowen; *Supertropical algebra*, Adv. in Math. 225(4) (2010), 2222-2286.
- Kontsevich M. and Y. Soibelman; *Affine structures and non-Archimedean analytic spaces*, In The unity of mathematics, volume 244 of Progr. Math., pages 321-385, Birkhäuser Boston, Boston, MA, 2006.
- Maclagan D. and B. Sturmfels, Tropical Geometry, preprint, 2009.
- Mikhalkin Grigory; *Tropical Geometry*, Book in progress. See http://www.math.toronto.edu/mikha/TG-project.html.
- Pin J.-E.; *Tropical semirings*, in: J. Gunawardena, (Ed.), Idempotency, Cambridge Univ. Press, 1998, pp. 50-60.
- Richter-Gebert J., B. Sturmfels, and T. Theobald; *First steps in tropical geometry. In Idempotent mathematics and mathematical physics*, volume 377 of Contemp. Math., pages 289-317, Amer. Math. Soc, 2005.
- Speyer David; *Tropical Geometry*, UC Berkeley, 2005.
- Viro O.; *Hyperfields for tropical geometry I*, Hyperfields and dequantization Preprint at arXiv:math.AG/1006.3034v2.
- Viro O.; *From the sixteenth Hilbert problem to tropical geometry*, Japan J. Math., 3 (2008) 1-30.

Tropical polynomials

Recall the definition of a tropical polynomial

$$\sum_{(i_1,\ldots,i_m)} a_{\mathbf{i}}\, x_1^{i_1} \ldots x_m^{i_m} \stackrel{\text{def}}{=} \sum_{\mathbf{i}} a_{\mathbf{i}}\, \mathbf{x}^{\mathbf{i}}.$$

Example

- $x^3 + 2x^2 + 6x + 11 = \min(3x, 2 + 2x, 6 + x, 11)$.
- $ax + by + c = \min(a + x, b + y, c)$.

Convention!

Violet means Tropical!

Roots of a tropical polynomial

Definition

- A root of a tropical polynomial $F(\mathbf{x}) = \sum_{\mathbf{i}} a_{\mathbf{i}}\, \mathbf{x}^{\mathbf{i}}$ is a point $\mathbf{x}_0$ such that the minimum is attained in at least two monomials.
- The set of roots is denoted by $V(F)$.

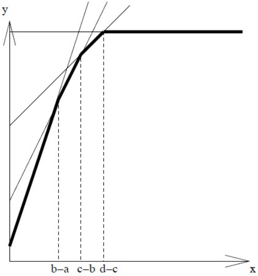
Example

$V(x^3 + 2x^2 + 6x + 11) = V(\min(3x, 2 + 2x, 6 + x, 11)) = \{2, 4, 5\}.$

Roots of a tropical polynomial: Example

Assume that $b - a \leq c - b \leq d - c$, then

$$ax^3 + bx^2 + cx + d = a(x + (b - a))(x + (c - b))(x + (d - c)).$$



Courtesy of D. Maclagan and B. Sturmfels 2013.

Roots of a tropical polynomial

Roots of an ordinary polynomial

What are the roots of $x^2 - x + t = t^0x^2 - t^0x + t^1$?

They are, $x_1 = \sum_{k=1}^{\infty} \frac{1}{k+1}\binom{2k}{k} t^k$, and $x_2 = 1 - x_1$.

Roots of the tropicalization

What are the roots of $x^2 + x + 1 = \min(0 + 2x, 0 + x, 1)$?

They are $x_1 = 1$, and $x_2 = 0$.

Field of Puisuex series, $P[t]$

Let $\mathbb{K}$ be an algebraic closed field of characteristic 0 (e.g. $\mathbb{C}$).

1. Elements:
$$p(t) = \sum_{\tau \in \mathbb{Q}_{\geq 0}, c_\tau \in \mathbb{K}} c_\tau t^\tau,$$
where summation is over well-ordered subsets of $\mathbb{Q}$.
2. The set of such series, $P[t]$, with natural operations is also an algebraic closed field.
3. What is the dominant term in $p(t)$ when $t \to 0$?

Order valuations

The order valuation ν on $P[t]$ is defined as

$$\nu(p) \stackrel{\text{def}}{=} \min\{\tau \in \mathbb{Q}_{\geq 0} \ : \ c_\tau \neq 0\}.$$

Note that,

1. $\nu(pq) = \nu(p) + \nu(q)$.
2. $\nu(p+q) \geq \min(\nu(p), \nu(q))$.
3. Note that when $t \to 0$ the dominant term in $p(t)$ is $c_{\nu(p(t))} t^{\nu(p(t))}$.

Kapranov, Mikhalkin, Rullgård

Kapranov, Mikhalkin, Rullgård

Given an ordinary polynomial $F(\mathbf{x}) = \sum_{\mathbf{i}} a_{\mathbf{i}}\, \mathbf{x}^{\mathbf{i}}$ with $a_{\mathbf{i}} \in P[t]$ and its tropicalization $\tilde{F}(\mathbf{x}) = \sum_{\mathbf{i}} \nu(a_{\mathbf{i}})\, \mathbf{x}^{\mathbf{i}}$, then $\nu(V(F)) = V(\tilde{F})$.

ν as a limit process!

Note that $\nu(a) = \lim_{t\to 0^+} \frac{\log |a(t)|}{\log t} = \lim_{t\to 0^+} \log_t |a(t)|$.

Example

Valuation of the roots

What are the roots of $x^2 - x + t = t^0 x^2 - t^0 x + t^1$?

They are, $\nu(x_1) = \nu\left(\sum_{k=1}^{\infty} \frac{1}{k+1}\binom{2k}{k} t^k\right) = 1$, and $\nu(x_2) = \nu(1 - x_1) = 0$.

Roots of the tropicalization

What are the roots of $x^2 + x + 1 = \min(0 + 2x, 0 + x, 1)$?

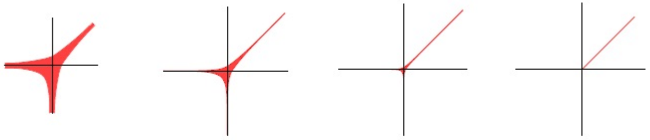
They are $x_1 = 1$, and $x_2 = 0$.

Amoebas

The Log map

$$\log_t : (\mathbb{C}^*)^2 \to \mathbb{R}^2, \quad -\log_t(x, y) = (-\log_t |x|, -\log_t |y|).$$

The amoebas $-\log_t(V(x + y + 1))$.



a) $\text{Log}(\mathcal{L})$ b) $\text{Log}_{t_1}(\mathcal{L})$ c) $\text{Log}_{t_2}(\mathcal{L})$ d) $\lim_{t\to\infty} \text{Log}_t(\mathcal{L})$

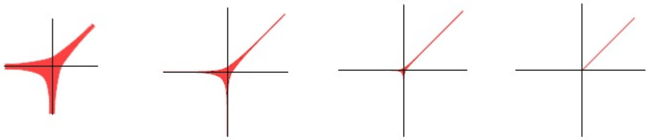
Courtesy of E. Brugallé 2013.

Amoebas

Note

$$\nu(V(x+y-1)) = \nu(V(t^0x+t^0y+t^0)) = V(x+y+0).$$

The amoebas $\log_t(V(x+y+0))$.



a) $\mathrm{Log}(\mathcal{L})$ b) $\mathrm{Log}_{t_1}(\mathcal{L})$ c) $\mathrm{Log}_{t_2}(\mathcal{L})$ d) $\lim_{t\to\infty}\mathrm{Log}_t(\mathcal{L})$

There is more!

For more on this see Bergman, Maslov, Mikhalkin, Passare and Rullgård, contributions on deformations and dequantization.

Why tropical algebra/geometry?

1. There is a nice geometry (e.g. a nice intersection theory).
2. Mikhalkin's approach to compute Gromov-Witten invariants through counting tropical curves. Also, Gathmann and Markwig's tropical/combinatorial proof of Kontsevich recursive formula for this.
3. Multivalued algebra and Viro's tropical groups.
4. Effective algorithms in tropical settings to compute important applied and theoretical problems (e.g. discriminants and geometry of polytops and varieties).

Tropical algebra/geometry is algebraic geometry in the limit!
This is our first example of this limit process!
We talk more about this!

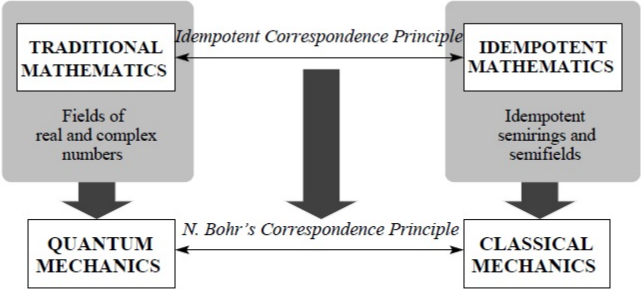
Idempotent Analysis, Dequantization and Correspondence Principle

Dequantization and Correspondence Principle

GENERAL REFERENCES FOR FURTHER READING:

- Baccelli F., G. Cohen, G. J. Olsder, and J.-P. Quadrat; *Synchronization and Linearity, An Algebra for Discrete Event Systems*, John Wiley and Sons, New York, New York, USA, 1992. Available at www-rocq.inria.fr/metalau/cohen/SED/SED1-book.html.
- Daneshgar A.; *Reconstruction in a generalized model for translation invariant systems*, Fuzzy Sets And Systems, 83 (1996), no. 1, 51–55.
- Gondran M. and M. Minoux; *Graphs, Dioids and Semirings*, Springer, New York, 2010.
- Kolokoltsov V. N. and V. P. Maslov; *Idempotent Analysis and Applications*, Kluwer Acad. Publ., Dordrecht, 1997.
- Lawson Jimmie D.; *Idempotent analysis and continuous semilattices*, Theoretical Computer Science, 316 (2004), 75–87.
- Litvinov Grigory L.; *Idempotent and tropical mathematics. Complexity of algorithms and interval analysis* , Available at arXiv:1209.1721.
- Litvinov Grigory L.; *Idempotent/tropical analysis, the Hamilton-Jacobi and Bellman equations* , Available at arXiv:1203.0522.
- Litvinov Grigory L.; *Tropical Mathematics, Idempotent Analysis, Classical Mechanics and Geometry* , Available at arXiv:1005.1247.
- Litvinov Grigory L., S. N. Sergeev (Eds.); *Tropical and Idempotent Mathematics*, Contemporary Mathematics, Vol. 495, AMS, Providence, RI, 2009.
- Litvinov Grigory L. and Victor P. Maslov (Eds.); *Idempotent mathematics and mathematical physics*, Contemporary Mathematics, Vol. 377, AMS, Providence, RI, 2005.
- Maslov Victor P. and S. N. Samborskii (Eds.); *Idempotent Analysis*, Adv. Soviet Math., 13, Amer. Math. Soc., Providence, R.I., 1992.

Correspondence principle



Litvinov's formulation for the limit process.

Let's delve into more details!

Courtesy of Litvinov 2010.

Maslov's dequantization

Deformation of $\mathbb{R}$

Consider $(\mathbb{R}^+, +, \times)$ and let h be a positive (i.e. Planck's) constant. Then define $(\mathbb{R}_h, \oplus_h, \otimes_h)$ to be the image of the following map for which the operations $\oplus_h$ and $\otimes_h$ are defined in such a way that make it a natural map, i.e.,

$$\pi_h : \mathbb{R}^+ \to \mathbb{R}_h \quad u \mapsto w = \pi_h(u) \stackrel{\text{def}}{=} h \ln u,$$

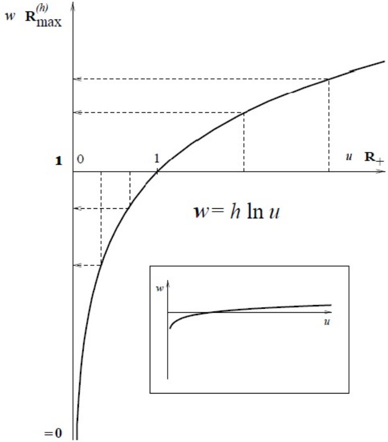
where

$$x \oplus_h y \stackrel{\text{def}}{=} h \ln(e^{\frac{x}{h}} + e^{\frac{y}{h}}),$$

and

$$x \otimes_h y \stackrel{\text{def}}{=} h \ln(e^{\frac{x}{h}} \times e^{\frac{y}{h}}) = x + y.$$

Maslov's dequantization map



Courtesy of Litvinov 2010.

Maslov's dequantization

Identities

Note that when $h \to 0^+$ then $\mathbb{R}_h \to \mathbb{R}_{\max} \stackrel{\text{def}}{=} (\mathbb{R} \cup \{-\infty\}, \max, +)$.
Clearly in this setting,

$$\pi(1) = 0 \quad \text{and} \quad \pi(0) = -\infty.$$

Aboebas

Note that a change of variable $h \to \frac{1}{\ln t}$ changes the map π to $\pi_t(u) = \nu(u) = \log_t u$ which is the amoebas map.

Hence, $\mathbb{R}^+$ can be thought of as a quantization of $\mathbb{R}_{\max}$!

Dynamics in the limit!

What happens if we dequantize classical dynamics?

Let's try to find the image in the limit under Maslov's dequantization.

Dequantization of the heat equation

Consider the heat equation and apply the transform $w = h \ln u$,

$$\frac{\partial u}{\partial t} = \frac{h}{2}\frac{\partial^2 u}{\partial x^2} \quad \mapsto \quad \frac{\partial w}{\partial t} - \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - \frac{h}{2}\frac{\partial^2 w}{\partial x^2} = 0,$$

and pass to the limit $h \to 0^+$ to get,

$$\mapsto \quad \frac{\partial w}{\partial t} = \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2,$$

which is (a special case) of the Hamilton-Jacobi-Bellman equation, with linear properties for solutions in $\mathbb{R}_{min}$.

The Brownian decision process I

Again consider the discrete time decision (control) process,

$$x(0) = x_0, \quad x(t+h) = x(t) - u(t),$$

along with the cost function

$$\min_u \left(\Phi(x(T)) + \sum_{i=0}^{T/h-1} \frac{u(ih)^2}{2h} \right).$$

This can also be solved through the dynamic programming equation

$$v(t,x) = \min_u \left(\frac{u^2}{2h} + v(t+h, x-u) \right), \quad v(T,.) = \Phi.$$

The Brownian decision process II

Use the change of control $u = hw$ in the dynamic programming equation and let $h \mapsto 0$ to get

$$\frac{\partial v}{\partial t} + \min_w \left(-w \frac{\partial v}{\partial x} + \frac{w^2}{2} \right) = 0, \quad v(T,.) = \Phi.$$

i.e. equivalent to the Hamilton-Jacobi-Bellman equation,

$$\frac{\partial v}{\partial t} = \frac{1}{2} \left(\frac{\partial v}{\partial x} \right)^2, \quad v(T,.) = \Phi.$$

note

In this sense the theory of Brownian decision processes is a discretized dequantizations of the theory of heat equations.

Dynamic programming I

Consider a very simple discrete decision (control) process,

$$x(0) = x_0, \quad x(n+1) = x(n) - u(n),$$

along with the cost function

$$Cost(N) = \min_{u(0),u(1),\ldots,u(N-1)} \left(\varphi(x(N)) + \sum_{i=0}^{N-1} c(u(i)) \right).$$

note

Functions c and φ satisfy typical conditions as being convex, lower-semicontinuous, and zero at their minimum, etc.

Inf-convolution of quadratic forms

Definition

The inf-convolution $f \boxtimes g$ is defined as $(f \boxtimes g)(z) \mapsto \inf_{x+y=z} f(x) + g(y)$.

Example: For any $m \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$ define

$$Q_{m,\sigma}(x) \stackrel{\text{def}}{=} \frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2 \quad \text{for } \sigma \neq 0,$$

$$Q_{m,0}(x) \stackrel{\text{def}}{=} \delta_m(x) \stackrel{\text{def}}{=} \begin{cases} 0 & x = m \\ +\infty & \text{otherwise.} \end{cases}$$

Then,

$$(Q_{m_1,\sigma_1} \boxtimes Q_{m_2,\sigma_2})(z) = Q_{m_1+m_2,\sqrt{\sigma_1+\sigma_2}}(z).$$

Dynamic programming II

Define

$$v(n,x) \stackrel{\text{def}}{=} \min_{u(n),u(1),\ldots,u(N-1)} \left(\varphi(x(N)) + \sum_{i=n}^{N-1} c(u(i)) \;\middle|\; x(n) = x \right), \text{ and}$$

note that it satisfies the dynamic programming equation

$$v(n,x) = \min_u \left(c(u) + v(n+1, x-u)\right), \quad v(N,x) = \varphi(x).$$

This equation can be written as follows using inf-convolution $\boxtimes$,

$$v(n,.) = c \boxtimes v(n+1,.), \quad v(N,.) = \varphi.$$

note

Hence, the solution is $v(0,.) = c^N \boxtimes \varphi$.

Discrete dynamics in the limit!

Hence dynamics in the discrete case is related to computing iterations (i.e. powers) of an operator.

Let's concentrate on this in the dequantized world mimicing linear operators!

Why mimicing Linear algebra in $\mathbb{R}_{max}$

A large number of results of conventional linear algebra can be extended to $\mathbb{R}_{max}$, and even to matrices over an arbitrary semiring, as

- The Cayley-Hamilton theorem.
- Determinants
- Polynomial functions
- Formal series

Semirings and graphs

Let S be a semiring and $A \in Mat_{n\times n}(S)$ be a matrix with entries in S. Then one may naturally identify A with a weighted directed graph structure, on n vertices, where there is a directed edge from the vertex u to the vertex v of weight $\epsilon \neq a_{uv} \in S$ if and only if this value is the entry in the row u and the column v of the matrix A.

note

Standard notions as paths, cycles and their weights are defined naturally.

The shortest path problem

definition

For any matrix A in $(D, \oplus, \otimes)$ we define A^* as follow

$$A^* \stackrel{\text{def}}{=} I \oplus A \oplus A^2 \oplus \cdots \oplus A^k \oplus \cdots$$

Questions

When does this limit A^* exists?
How can we compute it?

The shortest path problem

Let $S = \mathbb{R}_{min} \stackrel{\text{def}}{=} (\mathbb{R} \cup \{+\infty\}, \wedge, +)$ and $A \in Mat_{n\times n}(S)$ be the adjacency matrix of the weighted directed graph G on the vertex set $\{v_1, v_2, \ldots, v_n\}$. Then the shortest path problem is equal to find the matrix $D = [d_{ij}]$ such that d_{ij} is the minimum of the weights of all paths starting from v_i and ending at v_j. Note that in this setting we have,

$$D = A^* = I \wedge A \wedge A^2 \wedge \cdots \wedge A^k \wedge \cdots$$

Questions

When does this limit A^* exists?
How can we compute it?

A simple optimization problem

Let $S = \mathbb{R}_{max} \stackrel{\text{def}}{=} (\mathbb{R} \cup \{-\infty\}, \vee, +)$ and $A \in Mat_{n\times n}(S)$ be the adjacency matrix of the weighted directed graph G on the vertex set $\{v_1, v_2, \ldots, v_n\}$. Interpret a_{ij} as the profit of going from v_i to v_j and let $f_i \in \mathbb{R}$ be a terminal prize of ending at vertex v_i. Therefore, the solution of maximizing the income after k steps is equal to $D_k = A^k f$. Also, an overall maximizing solution is

$$D = A^* f.$$

Questions

When does this limit $A^* f$ exists?
How can we compute it?

Simple equations in $\mathbb{R}_{max}$

Let $\mathbb{R}_{max} \overset{\text{def}}{=} (\mathbb{R} \cup \{-\infty\}, \vee, +)$ and $A \in Mat_{n\times n}(\mathbb{R}_{max})$ be the adjacency matrix of the weighted directed graph G. Then, If there are only circuits of nonpositive weight in G, there is a solution to $x = Ax \vee b$ which is given by $x = A^*b$. Moreover, if the circuit weights are negative, the solution is unique.

note

Hint: $A(A^*b) \vee b = (e \vee AA^*)b = A^*b.$

Simple discrete dynamics in $\mathbb{R}_{max}$

Let $A \in Mat_{n\times n}(\mathbb{R}_{max})$ and consider the dynamics

$$x(n+1) = Ax(n), \quad x(0) = x_0,$$

whose solution is $x(n) = A^n x(0)$. Naturally, part of the analysis depend on the existence of eigenvalues and eigenfunctions for A, i.e. the existence of λ and f such that $Af = \lambda f$.

Eigenvalues and eigenvectors in $\mathbb{R}_{max}$

If $A \in Mat_{n\times n}(\mathbb{R}_{max})$ is irreducible, or equivalently if the corresponding graph G is strongly connected, there exists one and only one eigenvalue (but possibly several eigenvectors). This eigenvalue is equal to the maximum cycle mean of the graph G, i.e.

$$\lambda = \max_{c} \frac{|c|_w}{|c|_1},$$

where c ranges over cycles of the graph G.

Eigenvalues and eigenvectors (examples)

Example

Eigenfunctions are not unique. $e = 0, \epsilon = -\infty.$

$$\begin{pmatrix} 1 & e \\ e & 1 \end{pmatrix} \begin{pmatrix} e \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ e \end{pmatrix} = 1 \begin{pmatrix} e \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & e \\ e & 1 \end{pmatrix} \begin{pmatrix} -1 \\ e \end{pmatrix} = \begin{pmatrix} e \\ 1 \end{pmatrix} = 1 \begin{pmatrix} -1 \\ e \end{pmatrix}$$

Eigenvalues and eigenvectors (examples)

Example

Eigenvalues are not unique.

$e = 0, \epsilon = -\infty.$

$$\begin{pmatrix} 1 & \epsilon \\ \epsilon & 2 \end{pmatrix} \begin{pmatrix} e \\ \epsilon \end{pmatrix} = 1 \begin{pmatrix} e \\ \epsilon \end{pmatrix}$$

$$\begin{pmatrix} 1 & \epsilon \\ \epsilon & 2 \end{pmatrix} \begin{pmatrix} \epsilon \\ e \end{pmatrix} = 2 \begin{pmatrix} \epsilon \\ e \end{pmatrix}$$

Why idempotent analysis?

1. There is a nice functional analysis and operator theory (giving rise to a nonlinear system theory!).
2. A lot of linear algebra can be extended to this setting.
3. Again provides a nice example for the fact that there is a rich limit theory in which much of the facts reflects!

What are the properties of this new mathematics in the limit?!

A Nonlinear System Theory

References

General references for further reading:

- Daneshgar Amir; *Residuated semigroups and morphological aspects of translation invariant systems*, Fuzzy Sets And Systems, 90 (1997), no. 1, 69–81.
- Dubois Diedier and Henri Prade; *Toll Sets*, Proceedings of IFSA'91, International Fuzzy Systems Association, Brussels, 1991.
- Dress Andreas and Walter Wenzel ; *Algebraic, tropical, and fuzzy geometry*, Beitr. Algebra Geom., 52 (2011) 431–461.
- Gerla B.; *Many valued logics and semirings*, Neural Networks World, 5 (2003), 467–480.
- Joseph Goguen Joseph; *The logic of inexact concepts*, Synthese 19 (1968/9), 325-373.
- Litvinov Grigory L.; *Dequantization of Mathematics, idempotent semirings and fuzzy sets* , Available at arXiv:1209.1718.
- Litvinov Grigory L.; *Hypergroups and Hypergroup Algebras* , Available at arXiv:1109.6596.
- Wechler Wolfgang; *R-fuzzy computation*, J. Math. Anal. Appl. 115 (1986), 225-232.

Motvations

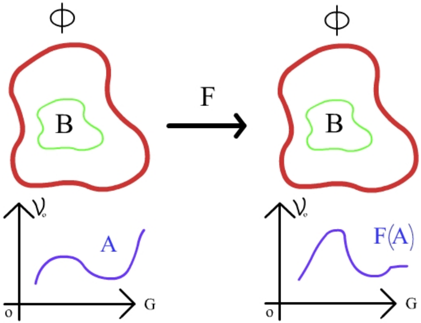
In this part we present an input/output system theory based on semigroups (actually dioids) in which the emphasis is on the role of signals as (fuzzy/toll) sets rather than functions. Hence, this is an example of a system theory based on the set-based approach.

Remark

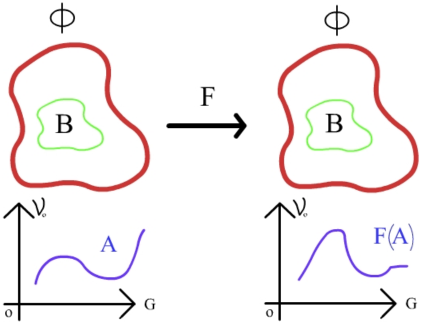
It is interesting if one can show that there exists a setup in which this is a limit theory of a classical case (e.g the theory of linear shift-invariant systems).

We will talk more about this!

Basic ideas for an I/O system theory

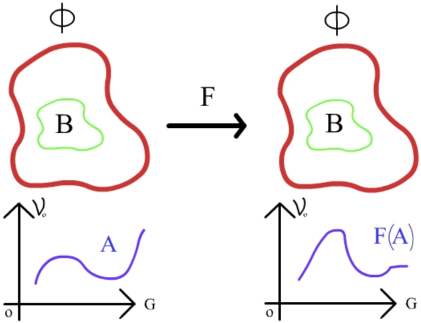


Basic ideas for an I/O system theory



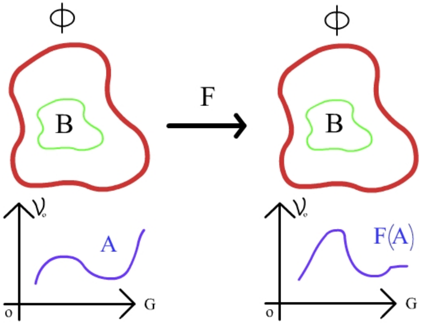
- Φ is a function space with some nice algebraic and topologic properties (usually completeness conditions).

Basic ideas for an I/O system theory



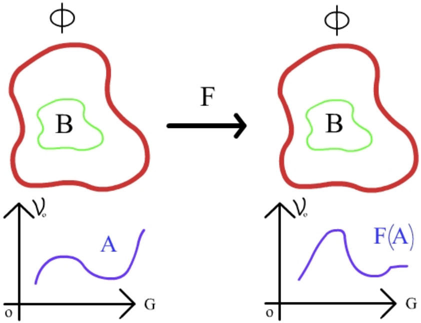
- Specially Φ is reconstructable, i.e. can be reconstructed properly by a (relatively small) subspace B of its generic objects.

BASIC IDEAS FOR AN I/O SYSTEM THEORY



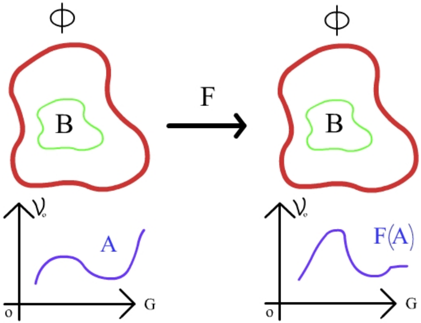
- Usually Φ is chosen to be a product space $G \times \mathcal{V}_0$.
  e.g. SPEACH: $G = R$ is the time and $\mathcal{V}_0 = R$ is the space of levels.
  e.g. IMAGE: $G = Z^2$ is the two dimensional space $\mathcal{V}_0 = Z$ is the space of gray-scales.

BASIC IDEAS FOR AN I/O SYSTEM THEORY



- *F* is a natural map (i.e compatible with the structure), with nice representation properties.

Basic ideas for an I/O system theory



- The whole setup should be in coherence with natural phenomena and be able to simulate input-output behaviour of such systems (this is called I/O system theory).

A CLASSICAL EXAMPLE: LSI SYSTEMS (OPERATORS)

- Φ is a Hilbert space (or some nice Sobolev space).

A CLASSICAL EXAMPLE: LSI SYSTEMS (OPERATORS)

- Φ is a Hilbert space (or some nice Sobolev space).
- F is linear and shift invariant, i.e

$$F(A + \alpha B) = F(A) + \alpha F(B),$$

$$F(T_g(A)) = T_g(F(A)),$$

where T_g is the translation-by-g operator, i.e.

$$T_g(A)(t) = A(t - g).$$

A CLASSICAL EXAMPLE: LSI SYSTEMS (OPERATORS)

- Φ is a Hilbert space (or some nice Sobolev space).
- F is linear and shift invariant, i.e

$$F(A + \alpha B) = F(A) + \alpha F(B),$$

$$F(T_g(A)) = T_g(F(A)),$$

where T_g is the translation-by-g operator, i.e.

$$T_g(A)(t) = A(t - g).$$

- We have the following reconstruction:

$$F(A)(t) = Conv(A, \delta_F) = \int_{-\infty}^{+\infty} \delta_F(\tau) A(t - \tau) d\tau.$$

LSI SYSTEMS (EXAMPLE)

Consider the discrete system

$$T(f)(n) \stackrel{\text{def}}{=} \frac{1}{3}(f(n) + f(n-1) + f(n-2))$$

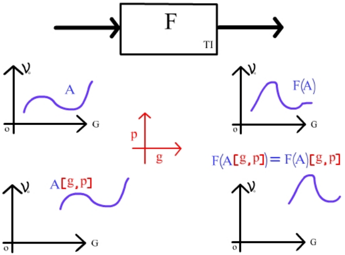
as a smoother (low-pass filter) on discrete signals $f : \mathbf{Z} \longrightarrow \mathbf{R}$. The first important observation is that T is an LSI system and its impulse response h is the following:

$$h(n) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{3} & n = 0, 1, 2 \\ 0 & n \neq 0, 1, 2. \end{cases}$$

Therefore, T can be expressed as the convolution

$$T(f)(n) = \sum_{m=0}^{\infty} f(n-m)h(m) = \frac{1}{3}(f(n) + f(n-1) + f(n-2)).$$

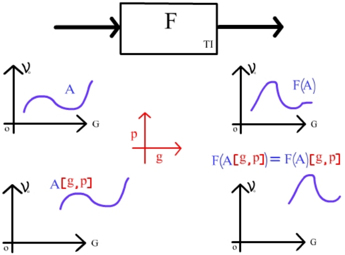
Translation invariance



- Consider

$$A = \{(t, A(t)) \mid t \in G\}.$$

TRANSLATION INVARIANCE



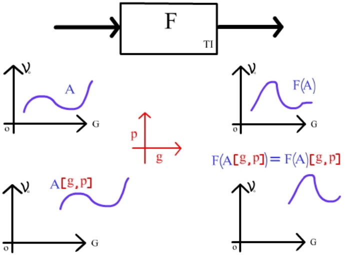
- Then we define the translation operator as follows,

$$A[g,p] = \{(t+g, A(t)*p) \mid t \in G\},$$

which means,

$$A[g,p](t) = A(t-g)*p.$$

TRANSLATION INVARIANCE



- An operator F is translation invariant if

$$F(A[g,p]) = F(A)[g,p].$$

Note: For an LSI operator, translation invariance on the range is equivalent to DC-gain 1.

OPERATORS V.S. COMPARISON

- The set $A = \{(t, A(t)) \mid t \in G\} \subseteq G \times \mathcal{V}_0$ can be considered as a function or a fuzzy set.

Operators v.s. Comparison

- The set $A = \{(t, A(t)) \mid t \in G\} \subseteq G \times \mathcal{V}_0$ can be considered as a function or a fuzzy set.
- The difference between the two points of view is in the way we look at the range $\mathcal{V}_0$.

Operators v.s. Comparison

- The set $A = \{(t, A(t)) \mid t \in G\} \subseteq G \times \mathcal{V}_0$ can be considered as a function or a fuzzy set.
- The difference between the two points of view is in the way we look at the range $\mathcal{V}_0$.
- The functional approach is when we consider algebraic properties and the fuzzy set approach is when we consider the comparative structure (order structure) of $\mathcal{V}_0$.
  e.g. The neutral element for summation is 0, while the neutral element for the supremum is $-\infty$.

MINKOWSKI ADDITION AND SUBTRACTION

- Let $(G, +, -, 0)$ be a group and $\mathcal{V}_0 = (\Omega, \leq, *, \div, 0) \cup \{-\infty, +\infty\}$ be a lattice ordered residuated semigroup with the universal bounds $-\infty$ and $+\infty$ such that,

$$\div(-\infty) = +\infty, \quad \div(+\infty) = -\infty,$$

$$(-\infty) * (+\infty) = (+\infty) * (-\infty) = (+\infty) * (+\infty) = (+\infty)$$

$$(-\infty) * (-\infty) = (-\infty),$$

$$\forall\, p \in \Omega \quad (+\infty) * p = p * (+\infty) = +\infty,$$

$$\forall\, p \in \Omega \quad (-\infty) * p = p * (-\infty) = -\infty.$$

Minkowski addition and subtraction

- Minkowski addition and subtraction for L-fuzzy sets are defined as follows

$$A \oplus B = \sup_g A[g, B(g)] \quad , \quad A \ominus B = \inf_g A[g, \div B(g)].$$

MINKOWSKI ADDITION AND SUBTRACTION

- Minkowski addition and subtraction for L-fuzzy sets are defined as follows

$$A \oplus B = \sup_g A[g, B(g)] \quad , \quad A \ominus B = \inf_g A[g, \div B(g)].$$

- e.g. Let
$G =< \ a \mid \ 3a = 0 \ >$,
$A = \{(0, x_1), (a, x_2), (2a, x_3)\}$, $B_1 = \{(0, 0), (a, 0), (2a, -\infty)\}$.
Then,

$$A \oplus B_1 = \{(0, \sup(x_1, x_3)), (a, \sup(x_2, x_1)), (2a, \sup(x_3, x_2))\}.$$

MINKOWSKI EROSION AS CONVOLUTION

- Minkowski erosion is defined as follows

$$Er(A, B) = A \ominus B^s.$$

where,

$$B^s = \{(-x, B(x)) \mid x \in G\}.$$

Minkowski erosion as convolution

- Minkowski erosion is defined as follows

$$Er(A,B) = A \ominus B^s.$$

where,

$$B^s = \{(-x, B(x)) \mid x \in G\}.$$

- Minkowski erosion is non-linear and behaves as a convolution operator.

The kernel

- For a TI operator F the kernel is defined as follows

$$K(F) = \{A \mid 0 \leq F(A)(0)\}.$$

THE KERNEL

- For a TI operator F the kernel is defined as follows

$$K(F) = \{A \mid 0 \leq F(A)(0)\}.$$

- The base of F is the set of all minimal elements of $K(F)$.

The kernel

- For a TI operator F the kernel is defined as follows

$$K(F) = \{A \mid 0 \leq F(A)(0)\}.$$

- The base of F is the set of all minimal elements of $K(F)$.
- e.g. Let
  $G =< \ a \mid \ 3a = 0 \ >$,
  $A = \{(0, x_1), (a, x_2), (2a, x_3)\}$, $F(A) = \{(0, m), (a, m), (2a, m)\}$,
  where $m = Median(x_1, x_2, x_3)$.
  Then,

$$B(F) = \{\{(0,0), (a,0), (2a, -\infty)\}, \{(0, -\infty), (a,0), (2a,0)\},$$

$$\{(0,0), (a, -\infty), (2a, 0)\}\}$$

THE RECONSTRUCTION THEOREM

- A TI operator F is isotone if

$$\forall\, A, B \;\; A \leq B \;\Rightarrow\; F(A) \leq F(B).$$

THE RECONSTRUCTION THEOREM

- A TI operator F is isotone if

$$\forall\, A, B \;\; A \leq B \;\Rightarrow\; F(A) \leq F(B).$$

- **Strong Reconstruction Theorem** (D. 1995)
  Let F be an isotone TI operator. Then

$$F(A) = \sup_{D \in K(F)} Er(A, D);$$

  and if the base of F exists then

$$F(A) = \sup_{B \in B(F)} Er(A, B).$$

Example 1: The mean filter

Again consider the discrete system

$$T(f)(n) \stackrel{\text{def}}{=} \frac{1}{3}(f(n)+f(n-1)+f(n-2)).$$

One may note that T is an isotone TI system with the basis consisting of all maps $h_{r,s}$, $(r,s \in \mathbf{R})$ defined as follows:

$$h_{r,s}(n) \stackrel{\text{def}}{=} \begin{cases} -r-s & n=-2 \\ s & n=-1 \\ r & n=0 \\ -\infty & otherwise. \end{cases}$$

$$\begin{aligned} T(f)(n) &= \sup_{r,s\in\mathbf{R}} Er(f,h_{r,s})(n) = \sup_{r,s\in\mathbf{R}} \inf_{m\in\mathbf{Z}} (f(n+m)-h_{r,s}(m)) \\ &= \sup_{r,s\in\mathbf{R}} \inf (f(n)-r, f(n-1)-s, f(n-2)+r+s). \end{aligned}$$

Example 2: The median filter

- Let $G =< \ a \mid \ 3a = 0 \ >$,
  $A = \{(0, x_1), (a, x_2), (2a, x_3)\}$, $F(A) = \{(0, m), (a, m), (2a, m)\}$,
  where $m = Median(x_1, x_2, x_3)$.
  Then,

$$B(F) = \{\{(0,0), (a,0), (2a, -\infty)\}, \{(0, -\infty), (a,0), (2a,0)\},$$

$$\{(0,0), (a, -\infty), (2a,0)\}\}$$

EXAMPLE 2: THE MEDIAN FILTER

- Let $G =< a \mid 3a = 0 >$,
  $A = \{(0, x_1), (a, x_2), (2a, x_3)\}$, $F(A) = \{(0, m), (a, m), (2a, m)\}$,
  where $m = Median(x_1, x_2, x_3)$.
  Then,

$$B(F) = \{\{(0,0), (a,0), (2a, -\infty)\}, \{(0, -\infty), (a, 0), (2a, 0)\},$$

$$\{(0,0), (a, -\infty), (2a, 0)\}\}$$

- Strong reconstruction theorem implies that
  $Median(x_1, x_2, x_3) = F(A)(0)$
  $= \sup(\inf(x_1, x_2), \inf(x_2, x_3), \inf(x_3, x_1))$.

Example 2: The median filter

- Let $G =< \; a \,|\; 3a = 0 \; >$,
  $A = \{(0, x_1), (a, x_2), (2a, x_3)\}$, $F(A) = \{(0, m), (a, m), (2a, m)\}$,
  where $m = Median(x_1, x_2, x_3)$.
  Then,

$$B(F) = \{\{(0,0), (a,0), (2a, -\infty)\}, \{(0, -\infty), (a, 0), (2a, 0)\},$$

$$\{(0,0), (a, -\infty), (2a, 0)\}\}$$

- Strong reconstruction theorem implies that
  $Median(x_1, x_2, x_3) = F(A)(0)$
  $= \sup(\inf(x_1, x_2), \inf(x_2, x_3), \inf(x_3, x_1))$.
- This is a fuzzification of the Boolean expression
  $x_1x_2 + x_2x_3 + x_3x_1$.

Sum up

- Reconstructions are usually as limits of convolutions.

SUM UP

- Reconstructions are usually as limits of convolutions.
- We know that,

SUM UP

- Reconstructions are usually as limits of convolutions.
- We know that,
  - Such limits can be defined in a variety of ways.

SUM UP

- Reconstructions are usually as limits of convolutions.
- We know that,
  - Such limits can be defined in a variety of ways.
  - Convolutions are essentially generalized HOM-functors.

Sum up

- Reconstructions are usually as limits of convolutions.
- We know that,
  - Such limits can be defined in a variety of ways.
  - Convolutions are essentially generalized HOM-functors.
  - Our general setup will cover both fuzzy and functional approach.

Thank you!

Comments and Criticisms are Welcomed

daneshgar@sharif.ir