Do Graphs Admit Topological Field Theories?

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Outline

What is a topological field theory?
- The category Cob(d+1)
- Catch a fleeting glimpse of Atiyah's TFT

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- What is a topological field theory?
  - The category Cob(d+1)
  - Catch a fleeting glimpse of Atiyah's TFT
- On the category of graphs
  - The homomorphism problem
  - Some computational facts

Outline

Basic ideas and motivations

- Basic physical motivation: Is it possible to construct a theory of everything? (in a physical sense!)
- This general unknown theory is usually referred to as quantum gravity.
- A breakthrough: Feynman's approach through action functionals and Feynman's integrals. (This does not cover Einstein's gravity!)
- One of the main ideas: Gluing dynamics on space-times together in a compatible way.
- Surprise: This gives rise to very interesting mathematical concepts and results too!

Atiyah's formulation

- In its simplest form, a $(d+1)$-dimensional TFT,

$$Z : \mathrm{Cob}(d+1) \to \mathrm{FVec}_K,$$

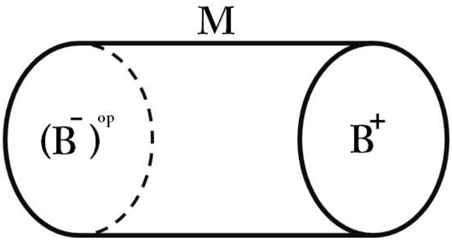
is a monoidal functor from the category $\mathrm{Cob}(d+1)$ to the category of finite dimensional vector spaces (over a fixed field K).

What does this mean?

Objects and 1-maps of $\mathrm{Cob}(d+1)$

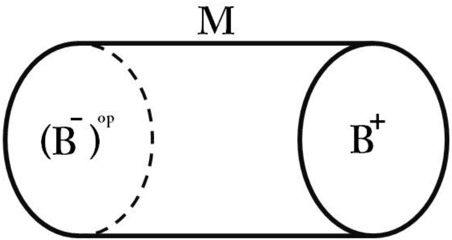
- The objects of $\mathrm{Cob}(d+1)$ are d-dimensional manifolds without boundary.
- Let B^- and B^+ be two such objects. Then a 1-map (or an ordinary map) is a diffeomorphism $f : B^- \to B^+$. This space is naturally equipped with the identity map and the composition of such maps.

Cobordisms and 2-maps in $\mathrm{Cob}(d+1)$



- A cobordism in $\mathrm{Cob}(d+1)$ is an equivalence class of (smooth, compact and oriented) manifolds, as $[M] : B^- \to B^+$, of dimension $(d+1)$ with boundary $\partial M = (B^-)^{op} \cup B^+$.
- Two such cobordisms, M and N are equivalent if there is a diffeomorphism $f : M \to N$ that restricts to identity map on their common boundary.

COBORDISMS AND 2-MAPS IN $\mathrm{Cob}(d+1)$



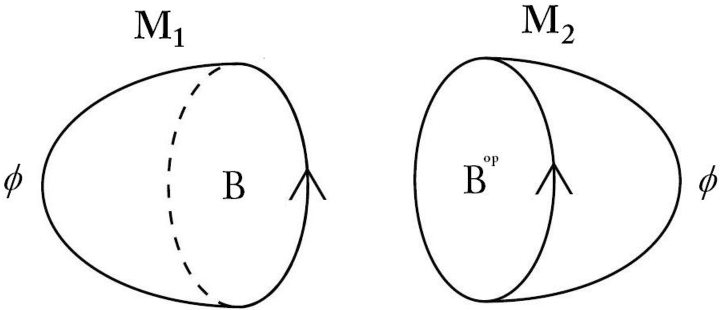
- A cobordism is a model of a space-time.
- Composition of cobordisms are defined by gluing!
- 2-morphisms between cobordisms are defined naturally!

$\mathrm{Cob}(d+1)$ AND FVec_K AS MONOIDAL CATEGORIES

- The category $\mathrm{Cob}(d+1)$ has a monoidal structure with respect to disjoint union of cobordisms.

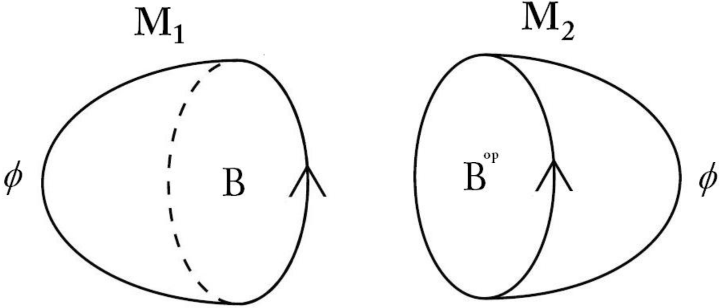
What does this mean?

Invariants through TFT's



- Identify $V^* \simeq \mathrm{Hom}(\mathrm{V}, \mathrm{K})$.
- Identify a $(d+1)$-manifold M with boundary ∂M with a cobordism $\emptyset \to \partial M$.
- Identify a $(d+1)$-manifold M^{op} with boundary ∂M^{op} with a cobordism $\partial M^{op} \to \emptyset$.

Invariants through TFT's



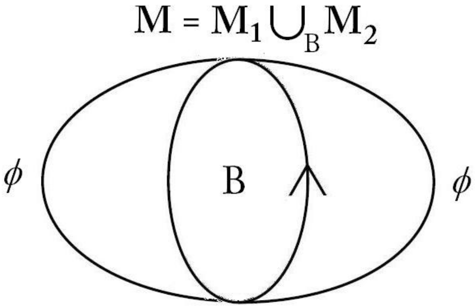
- Then applying Z we have:

$$Z(M) \equiv (f : Z(\emptyset) = K \to Z(\partial M)) \equiv (z(M) \stackrel{\text{def}}{=} f(1) \in Z(\partial M)).$$

- Note that $z(M^{op})$ can be identified with a map $z^{op}(M) \stackrel{\text{def}}{=} (Z(\partial M) \to Z(\emptyset) = K)$ as an element of $Z(\partial M)^*$.

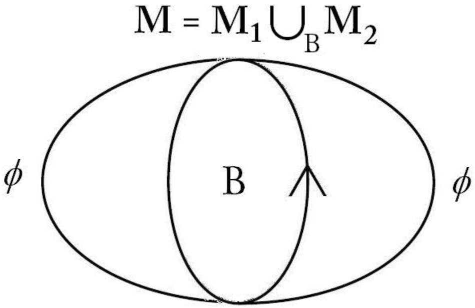
How one may compute $z(M)$?

Invariants through surgery I



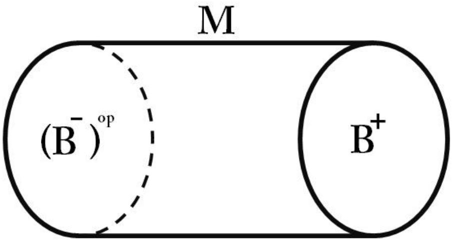
- Denote the duality in FVec_K by $\langle .,. \rangle : V^* \otimes V \to K$.
- Let a $(d+1)$-manifold M be cut through an embedded d-manifold B into two parts M_1 and M_2, i.e. $M = M_1 \cup_B M_2$ with $\partial M_1 = B$ and $\partial M_2 = B^{op}$.

Invariants through surgery I



- Thinking of M_1 and M_2 as cobordisms we have $Z(M) = Z(M_1) \circ Z(M_2)$ and consequently an element $z(M) = \langle z^{op}(M_2), z(M_1) \rangle$.
- Note that when $\partial M = \emptyset$ then through identification we have the numerical invariant $z^{op}(M) \equiv z(M) \in K$.

Invariants through surgery II

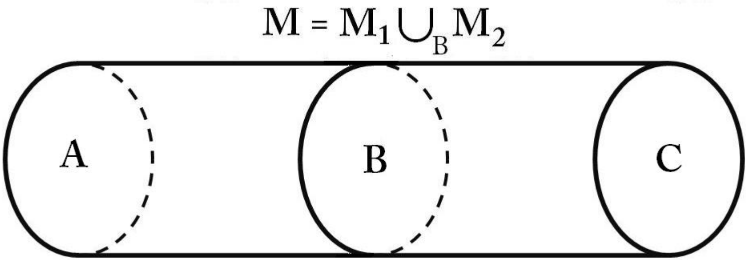


- More generally if $\partial M = (B^-)^{op} \cup B^+$ then

$$Z(\partial M) = Z((B^-)^{op} \cup B^+) \simeq Z(B^-)^* \otimes Z(B^+) \simeq$$

$$\simeq \mathrm{Hom}(Z(B^-), Z(B^+)).$$

Invariants through surgery II



- Hence for a gluing $M = M_1 \cup_B M_2$ with $\partial M_1 = A \cup B$ and $\partial M_2 = B^{op} \cup C$, we have,

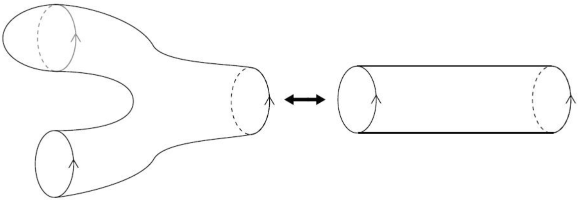
$$Z(M) = Z(M_2) \circ Z(M_1).$$

Invariants through surgery II

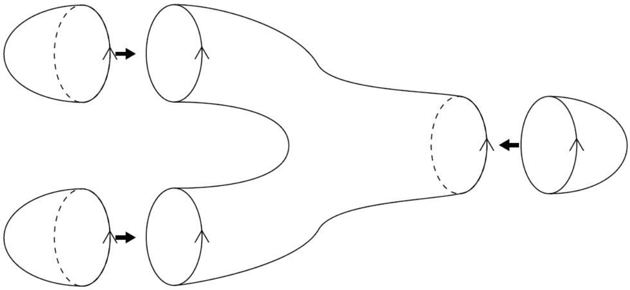
- Idea: One may obtain invariants of closed manifolds and categorize them if one can construct all such manifolds though amalgams of a finite number of basic atomics ones.

Let us see a couple of examples in dimension $1+1$!

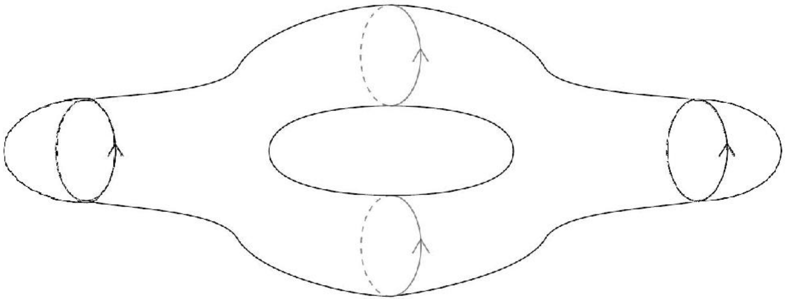
A CYLINDER



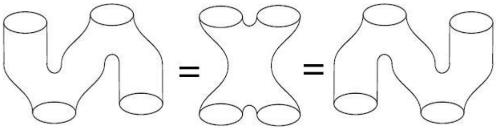
A SPHERE



A TORUS



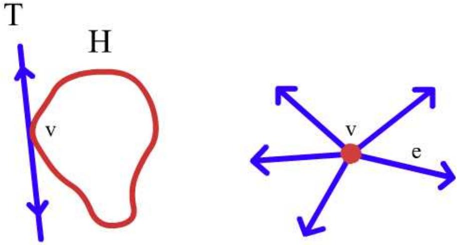
Frobenius relation



The Frobenius Relation

- There is an equivalence of categories between the category of 2-dimensional topological (quantum) field theories and commutative Frobenius algebras.

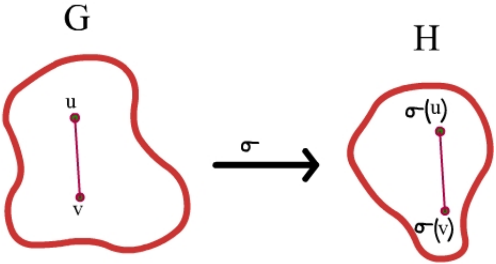
The tangent space



Continuous Case   Discrete Case

- The tangent space at vertex v is the set of out-going vectors from v!

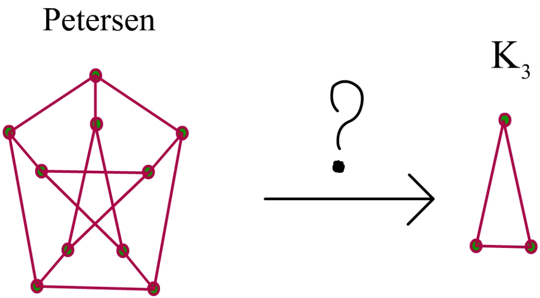
A GRAPH HOMOMORPHISM



A graph homomorphism

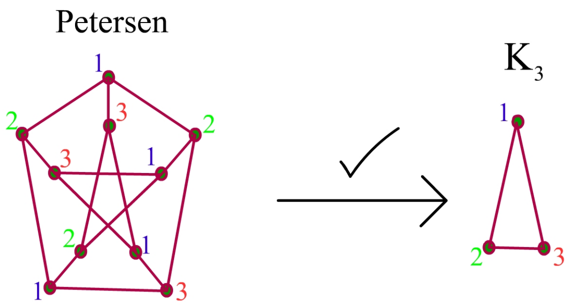
- A graph homomorphism σ from a graph G to a graph H is a map $\sigma : V(G) \longrightarrow V(H)$ such that $uv \in E(G)$ implies $\sigma(u)\sigma(v) \in E(H)$.

A QUESTION



- Does there exist a homomorphism from the Petersen graph to the triangle K_3?

Graph coloring



- Homomorphisms to K_n is equivalent to coloring the vertices of the graph by n colors such that the terminal ends of each edge have different colors.

The homomorphism problem I

- Fix a graph H and ask: Given a graph G, is it true that $\mathrm{Hom}(G, H) \neq \emptyset$.
- It is known that this decision problem is NP-complete in most symmetric cases of H, as for simple graphs and vertex-transitive directed graphs.

The homomorphism problem II

- It is known that any CSP reduced to a directed homomorphism problem.
- The dichotomy conjecture for CSP's imply that the homomorphism problem of graphs is either in P or is NP-complete.

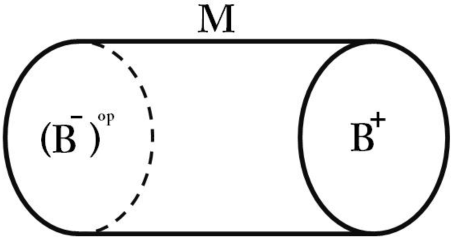
How can we tackle this conjecture?

Graphical TFT's

- Edges are counterparts of vector spaces (tangent spaces) in graphs.
- Hence, in order to mimic the scenario of a TFT in graph theory we ought to ask about a nice category of graphical cobordisms (i.e. cylinders)!

In what follows we investigate the existence of such a category of cylinders!

Graphical cylinders



- Intuitively a cylinder is a graph M with boundary $\partial M = B^{-} \cup B^{+}$.

Cylindrical and exponential constructions

- Given a cylinder C, there exist a cylindrical construction $G \boxtimes C$ and an exponential construction $[C, H]$, which are functorial and moreover,
  a) $\mathrm{Hom}(G \boxtimes C, H) \neq \emptyset \quad \Leftrightarrow \quad \mathrm{Hom}(G, [C, H]) \neq \emptyset.$
  b) There exist a retraction

$$r_{G,H} : \mathrm{Hom}(G \boxtimes C, H) \to \mathrm{Hom}(G, [C, H])$$

  and a section

$$s_{G,H} : \mathrm{Hom}(G, [C, H]) \to \mathrm{Hom}(G \boxtimes C, H)$$

  such that $r_{G,H} \circ s_{G,H} = \mathbf{1}$, where $\mathbf{1}$ is the identity mapping.

An example

- $\mathrm{Hom}(C_n \square K_2, K_3) \neq \emptyset \quad \Leftrightarrow \quad \mathrm{Hom}(C_n, [K_2 \square K_2, K_3]) \neq \emptyset.$

Some examples

A directed cylindrical construction	Show(C_{12})
Cartesian product as a cylindrical construction	Show($C_3 \square P_2$)
The Petersen graph	Show
Subdivision and powers	Show
The exponential graph $[K_2 \square K_2, K_3]$	Show

Epilogue

- The problem of defining a well-behaved category of graphical cylinders (cobordisms) and corresponding TFT's is not only important from a mathematical point of view but also is closely linked to fundamental problems in computer science as $P \stackrel{?}{=} NP$.
- It seems that such constructions may lead to situations which are quite similar (but maybe weaker than) Tensor-Hom duality which may be of independent interest in graph theory.

THE END.

Thank You!