Dioids, Systems and Categories

Amir Daneshgar

Department of Mathematical Sciences
Sharif University of Technology

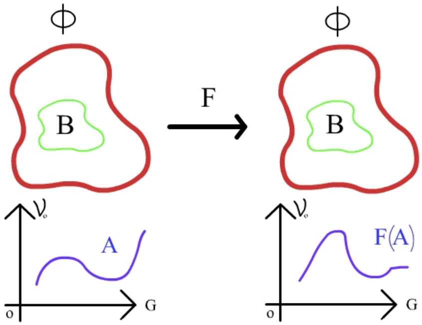
June 14, 2011

Outline

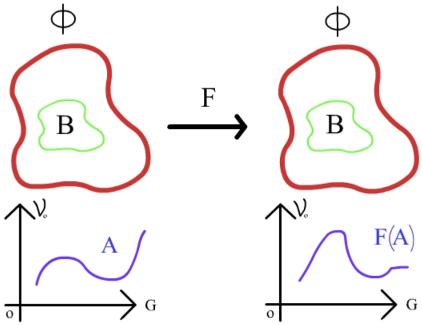
Outline

1. Introduction to I/O system theory
2. Fuzzy sets, dioids and dynamics

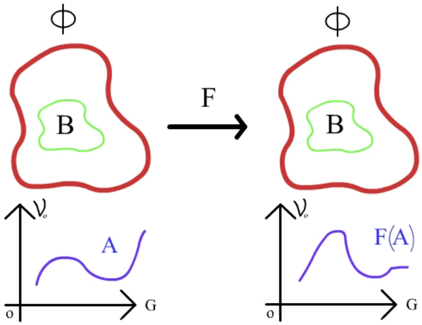
Basic idea



Basic Idea

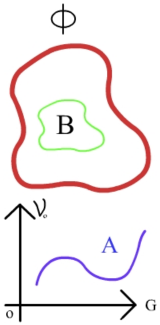


- Φ is a function space with some nice algebraic and topologic properties (usually completeness conditions).



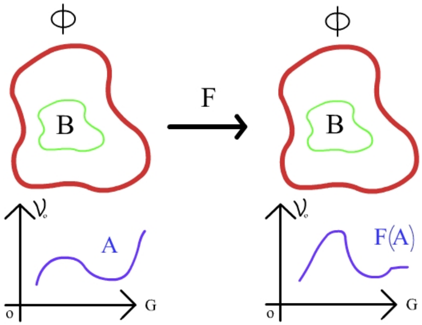
- Specially Φ is reconstructable, i.e. can be reconstructed properly by a (relatively small) subspace *B* of its generic objects.

BASIC IDEA



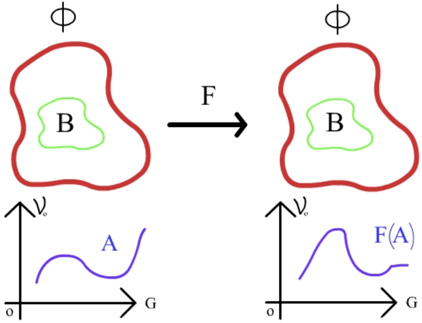
- Usually Φ is chosen to be a product space $G \times \mathcal{V}_0$.
  e.g. SPEACH: $G = R$ is the time and $\mathcal{V}_0 = R$ is the space of levels.
  e.g. IMAGE: $G = Z^2$ is the two dimensional space $\mathcal{V}_0 = Z$ is the space of gray-scales.

BASIC IDEA



- *F* is a natural map (i.e compatible with the structure), with nice representation properties.

Basic Idea



- The whole setup should be in coherence with natural phenomena and be able to simulate input-output behaviour of such systems (this is called I/O system theory).

A classical example: LSI systems (operators)

- Φ is a Hilbert space (or some nice Sobolev space).

A classical example: LSI systems (operators)

- Φ is a Hilbert space (or some nice Sobolev space).
- F is linear and shift invariant, i.e

$$F(A + \alpha B) = F(A) + \alpha F(B),$$

$$F(T_g(A)) = T_g(F(A)),$$

where T_g is the translation-by-g operator, i.e.

$$T_g(A)(t) = A(t - g).$$

A classical example: LSI systems (operators)

- Φ is a Hilbert space (or some nice Sobolev space).
- F is linear and shift invariant, i.e

$$F(A + \alpha B) = F(A) + \alpha F(B),$$

$$F(T_g(A)) = T_g(F(A)),$$

where T_g is the translation-by-g operator, i.e.

$$T_g(A)(t) = A(t - g).$$

- We have the following reconstruction:

$$F(A)(t) = Conv(A, \delta_F) = \int_{-\infty}^{+\infty} \delta_F(\tau) A(t - \tau) d\tau.$$

LSI SYSTEMS (EXAMPLE)

Consider the discrete system

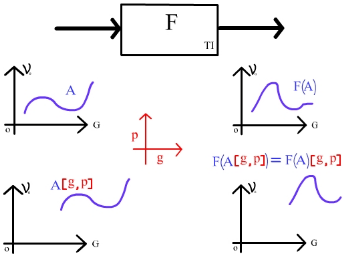
$$T(f)(n) \stackrel{\text{def}}{=} \frac{1}{3}(f(n) + f(n-1) + f(n-2))$$

as a smoother (low-pass filter) on discrete signals $f : \mathbf{Z} \longrightarrow \mathbf{R}$. The first important observation is that T is an LSI system and its impulse response h is the following:

$$h(n) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{3} & n = 0, 1, 2 \\ 0 & n \neq 0, 1, 2. \end{cases}$$

Therefore, T can be expressed as the convolution

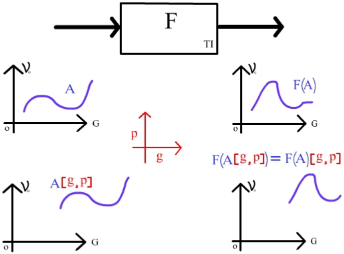
$$T(f)(n) = \sum_{m=0}^{\infty} f(n-m)h(m) = \frac{1}{3}(f(n) + f(n-1) + f(n-2)).$$



- Consider

$$A = \{(t, A(t)) \mid t \in G\}.$$

Translation Invariance



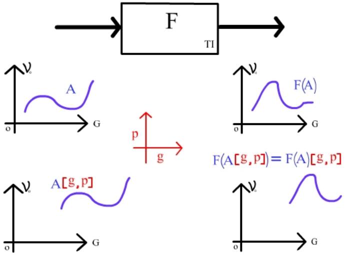
- Then we define the translation operator as follows,

$$A[g,p] = \{(t+g, A(t)*p) \mid t \in G\},$$

which means,

$$A[g,p](t) = A(t-g)*p.$$

Translation Invariance



- An operator F is translation invariant if

$$F(A[g,p]) = F(A)[g,p].$$

Note: For an LSI operator, translation invariance on the range is equivalent to DC-gain 1.

OPERATORS V.S. COMPARISON

- The set $A = \{(t, A(t)) \mid t \in G\} \subseteq G \times \mathcal{V}_0$ can be considered as a function or a fuzzy set.

Operators v.s. Comparison

- The set $A = \{(t, A(t)) \mid t \in G\} \subseteq G \times \mathcal{V}_0$ can be considered as a function or a fuzzy set.
- The difference between the two points of view is in the way we look at the range $\mathcal{V}_0$.

Operators v.s. Comparison

- The set $A = \{(t, A(t)) \mid t \in G\} \subseteq G \times \mathcal{V}_0$ can be considered as a function or a fuzzy set.
- The difference between the two points of view is in the way we look at the range $\mathcal{V}_0$.
- The functional approach is when we consider algebraic properties and the fuzzy set approach is when we consider the comparative structure (order structure) of $\mathcal{V}_0$.
  e.g. The neutral element for summation is 0, while the neutral element for the supremum is $-\infty$.

- Let $(G, +, -, 0)$ be a group and $\mathcal{V}_0 = (\Omega, \leq, *, \div, 0) \cup \{-\infty, +\infty\}$ be a lattice ordered group with the universal bounds $-\infty$ and $+\infty$ such that,

$$\div(-\infty) = +\infty, \quad \div(+\infty) = -\infty,$$

$$(-\infty) * (+\infty) = (+\infty) * (-\infty) = (+\infty) * (+\infty) = (+\infty)$$

$$(-\infty) * (-\infty) = (-\infty),$$

$$\forall\, p \in \Omega \quad (+\infty) * p = p * (+\infty) = +\infty,$$

$$\forall\, p \in \Omega \quad (-\infty) * p = p * (-\infty) = -\infty.$$

MINKOWSKI ADDITION AND SUBTRACTION

- Minkowski addition and subtraction for L-fuzzy sets are defined as follows

$$A \oplus B = \sup_g A[g, B(g)] \quad , \quad A \ominus B = \inf_g A[g, \div B(g)].$$

- Minkowski addition and subtraction for L-fuzzy sets are defined as follows

$$A \oplus B = \sup_g A[g, B(g)] \quad , \quad A \ominus B = \inf_g A[g, \div B(g)].$$

- e.g. Let
  $G =< \ a \mid \ 3a = 0 \ >$,
  $A = \{(0, x_1), (a, x_2), (2a, x_3)\}$,
  $B_1 = \{(0, 0), (a, 0), (2a, -\infty)\}$.
  Then,

  $$A \oplus B_1 = \{(0, \sup(x_1, x_3)), (a, \sup(x_2, x_1)), (2a, \sup(x_3, x_2))\}.$$

Minkowski erosion as convolution

- Minkowski erosion is defined as follows

$$Er(A, B) = A \ominus B^s.$$

where,

$$B^s = \{(-x, B(x)) \mid x \in G\}.$$

MINKOWSKI EROSION AS CONVOLUTION

- Minkowski erosion is defined as follows

$$Er(A, B) = A \ominus B^s.$$

where,

$$B^s = \{(-x, B(x)) \mid x \in G\}.$$

- Minkowski erosion is non-linear and behaves as a convolution operator.

The kernel

- For a TI operator F the kernel is defined as follows

$$K(F) = \{A \mid 0 \leq F(A)(0)\}.$$

The kernel

- For a TI operator F the kernel is defined as follows

$$K(F) = \{A \mid 0 \leq F(A)(0)\}.$$

- The base of F is the set of all minimal elements of $K(F)$.

The kernel

- For a TI operator F the kernel is defined as follows

$$K(F) = \{A \mid 0 \leq F(A)(0)\}.$$

- The base of F is the set of all minimal elements of $K(F)$.
- e.g. Let
  $G =< \ a \mid \ 3a = 0 \ >$,
  $A = \{(0, x_1), (a, x_2), (2a, x_3)\}$,
  $F(A) = \{(0, m), (a, m), (2a, m)\}$, where
  $m = Median(x_1, x_2, x_3)$.
  Then,

$$B(F) = \{\{(0,0), (a,0), (2a,-\infty)\}, \{(0,-\infty), (a,0), (2a,0)\}, \{(0,0), (a,-\infty), (2a,0)\}\}$$

The Reconstruction Theorem

- A TI operator F is isotone if

$$\forall\, A, B \;\; A \leq B \;\Rightarrow\; F(A) \leq F(B).$$

- A TI operator F is isotone if

$$\forall\, A, B \;\; A \leq B \;\Rightarrow\; F(A) \leq F(B).$$

- **Strong Reconstruction Theorem** (D. 1995)
  Let F be an isotone TI operator. Then

$$F(A) = \sup_{D \in K(F)} Er(A, D);$$

  and if the base of F exists then

$$F(A) = \sup_{B \in B(F)} Er(A, B).$$

EXAMPLE 1: THE MEAN FILTER

Again consider the discrete system

$$T(f)(n) \stackrel{\text{def}}{=} \frac{1}{3}(f(n) + f(n-1) + f(n-2)).$$

One may note that T is an isotone TI system with the basis consisting of all maps $h_{r,s}$, $(r, s \in \mathbf{R})$ defined as follows:

$$h_{r,s}(n) \stackrel{\text{def}}{=} \begin{cases} -r-s & n=-2 \\ s & n=-1 \\ r & n=0 \\ -\infty & \textit{otherwise.} \end{cases}$$

Hence,

$$\begin{aligned} T(f)(n) &= \sup_{r,s\in\mathbf{R}} Er(f, h_{r,s})(n) \\ &= \sup_{r,s\in\mathbf{R}} \inf_{m\in\mathbf{Z}} (f(n+m) - h_{r,s}(m)) \end{aligned}$$

Example 2: The median filter

- Let $G =< a \mid 3a = 0 >$,
  $A = \{(0, x_1), (a, x_2), (2a, x_3)\}$,
  $F(A) = \{(0, m), (a, m), (2a, m)\}$, where
  $m = Median(x_1, x_2, x_3)$.
  Then,

$$B(F) = \{\{(0,0), (a,0), (2a, -\infty)\}, \{(0, -\infty), (a, 0), (2a, 0)\},$$

$$\{(0,0), (a, -\infty), (2a, 0)\}\}$$

Example 2: The median filter

- Let $G =< \ a \mid \ 3a = 0 \ >$,
  $A = \{(0, x_1), (a, x_2), (2a, x_3)\}$,
  $F(A) = \{(0, m), (a, m), (2a, m)\}$, where
  $m = Median(x_1, x_2, x_3)$.
  Then,

$$B(F) = \{\{(0,0),(a,0),(2a,-\infty)\},\{(0,-\infty),(a,0),(2a,0)\},$$

$$\{(0,0),(a,-\infty),(2a,0)\}\}$$

- Strong reconstruction theorem implies that
  $Median(x_1, x_2, x_3) = F(A)(0)$
  $= \sup(\inf(x_1, x_2), \inf(x_2, x_3), \inf(x_3, x_1))$.

Example 2: The median filter

- Let $G = < \ a \mid \ 3a = 0 \ >$,
  $A = \{(0, x_1), (a, x_2), (2a, x_3)\}$,
  $F(A) = \{(0, m), (a, m), (2a, m)\}$, where
  $m = Median(x_1, x_2, x_3)$.
  Then,

$$B(F) = \{\{(0,0),(a,0),(2a,-\infty)\},\{(0,-\infty),(a,0),(2a,0)\},$$

$$\{(0,0),(a,-\infty),(2a,0)\}\}$$

- Strong reconstruction theorem implies that
  $Median(x_1, x_2, x_3) = F(A)(0)$
  $= \sup(\inf(x_1, x_2), \inf(x_2, x_3), \inf(x_3, x_1))$.
- This is a fuzzification of the Boolean expression
  $x_1 x_2 + x_2 x_3 + x_3 x_1$.

SUM UP

- Reconstructions are usually as limits of convolutions.

Sum up

- Reconstructions are usually as limits of convolutions.
- In what follows we show that

SUM UP

- Reconstructions are usually as limits of convolutions.
- In what follows we show that
    - Such limits can be defined in a variety of ways.

Sum up

- Reconstructions are usually as limits of convolutions.
- In what follows we show that
  - Such limits can be defined in a variety of ways.
  - Convolutions are essentially generalized HOM-functors.

Sum up

- Reconstructions are usually as limits of convolutions.
- In what follows we show that
  - Such limits can be defined in a variety of ways.
  - Convolutions are essentially generalized HOM-functors.
  - Our general setup will cover both fuzzy and functional approach.

Categorical prerequisites I

- Let $\mathcal{V} = (\mathcal{V}, ., \div, I, a, l, r, c)$ be a symmetric closed monoidal category enriched over itself, with identity I, for which $V.- \dashv - \div V$ for any $A \in V$ and the base category $\mathcal{V}_0$ is complete and cocomplete.

Categorical prerequisites I

- Let $\mathcal{V} = (\mathcal{V}, ., \div, I, a, l, r, c)$ be a symmetric closed monoidal category enriched over itself, with identity I, for which $V.- \dashv - \div V$ for any $A \in V$ and the base category $\mathcal{V}_0$ is complete and cocomplete.
- We also assume that V $= [I, -]_{\mathcal{V}_0} : \mathcal{V}_0 \longrightarrow$ Set is the base functor and we adapt the multiplicative notation in $\mathcal{V}$.

CATEGORICAL PREREQUISITES I

- Let $\mathcal{V} = (\mathcal{V}, ., \div, I, a, l, r, c)$ be a symmetric closed monoidal category enriched over itself, with identity I, for which $V.- \dashv - \div V$ for any $A \in V$ and the base category $\mathcal{V}_0$ is complete and cocomplete.
- We also assume that $\mathrm{V} = [I, -]_{\mathcal{V}_0} : \mathcal{V}_0 \longrightarrow \mathrm{Set}$ is the base functor and we adapt the multiplicative notation in $\mathcal{V}$.
- On the other hand, let $\mathcal{D}$ be a class of diagram–schemes and assume that $\mathcal{V}_f \subseteq \mathcal{V}_0$ is a small full subcategory such that $\mathcal{V}_0$ is a free $\mathcal{D}$–cocompletion of $\mathcal{V}_f$ in the sense that,
    - The totality of all $\mathcal{D}$–colimits constitute a density presentation for the inclusion $\mathrm{i} : \mathcal{V}_f \hookrightarrow \mathcal{V}_0$.
    - For any $A \in \mathcal{V}_f$, the hom–functor $[A, -]$ preserves all $\mathcal{D}$–colimits.

CATEGORICAL PREREQUISITES II

- we consider a (small) commutative group $(G, +, -)$, with identity 0 and we focus on the product space $\Phi = \mathcal{V}_0^G$ equipped with the pointwise categorical structure of $\mathcal{V}_0$.

Categorical prerequisites II

- we consider a (small) commutative group $(G, +, -)$, with identity 0 and we focus on the product space $\Phi = \mathcal{V}_0^G$ equipped with the pointwise categorical structure of $\mathcal{V}_0$.
- It is easy to see that, considering Φ as a category of functors between the discrete category G and $\mathcal{V}_0$, Φ inherits the completeness properties of $\mathcal{V}_0$ and that it has a dense $\mathcal{D}$–presentable full subcategory Φ_f.

- we consider a (small) commutative group $(G, +, -)$, with identity 0 and we focus on the product space $\Phi = \mathcal{V}_0^G$ equipped with the pointwise categorical structure of $\mathcal{V}_0$.
- It is easy to see that, considering Φ as a category of functors between the discrete category G and $\mathcal{V}_0$, Φ inherits the completeness properties of $\mathcal{V}_0$ and that it has a dense $\mathcal{D}$–presentable full subcategory Φ_f.
- Hereafter, $\Delta_W \in \Phi$ is the constant functor with value W.

- Consider the following maps for a fixed $D \in \Phi$,

$$\mathrm{T}_D : \mathrm{obj}(\Phi) \longrightarrow \mathrm{obj}(\Phi),$$

$$\mathrm{T}_D(A)_{(z)} = \prod_{|J|<\infty} \left(\bigoplus_{j \in J} W_j\right)^{[A_{(x)}, \oplus_j (\Delta_{W_j} \div D_{(-x+z)})]_\Phi},$$

$$\mathrm{H}_D : \mathrm{obj}(\Phi) \longrightarrow \mathrm{obj}(\Phi),$$

$$\mathrm{H}_D(B)_{(z)} = \coprod_{|J|<\infty}^{[\oplus_j (\Delta_{V_j} \cdot D_{(-z+x)}), B_{(x)}]_\Phi} \left(\bigoplus_{j \in J} V_j\right),$$

where $z \in G$ is a fixed coordinate and $\bigoplus$ is a suitable associative bifunctor.

Two functors

- Consider the following maps for a fixed $D \in \Phi$,
  $\mathrm{T}_D : \mathrm{obj}(\Phi) \longrightarrow \mathrm{obj}(\Phi)$,
  $$\mathrm{T}_D(A)_{(z)} = \prod_{|J|<\infty} (\bigoplus_{j \in J} W_j)^{[A_{(x)}, \oplus_j(\Delta_{W_j} \div D_{(-x+z)})]_\Phi},$$
  $\mathrm{H}_D : \mathrm{obj}(\Phi) \longrightarrow \mathrm{obj}(\Phi)$,
  $$\mathrm{H}_D(B)_{(z)} = \coprod_{|J|<\infty}^{[\oplus_j(\Delta_{V_j} \cdot D_{(-z+x)}), B_{(x)}]_\Phi} (\bigoplus_{j \in J} V_j),$$
  where $z \in G$ is a fixed coordinate and $\bigoplus$ is a suitable associative bifunctor.
- **Theorem**(D.& Hashemi 2000)
  Both T_D and H_D can be naturally extended to define functors on Φ for any $D \in \Phi$.

AN IMPORTANT SPECIAL CASE

- If $\bigoplus$ is preserved by $(\div)$ (resp. $(.)$) then

$$\mathrm{T}_D(A)_{(z)} = \coprod_{x \in G} A_{(x)} . D_{(-x+z)} \stackrel{\text{def}}{=} \dot{\mathrm{T}}_D(A)_{(z)}$$

$$(\textit{resp.}\ \ \mathrm{H}_D(B)_{(z)} = \prod_{x \in G} B_{(x)} \div D_{(-z+x)} \stackrel{\text{def}}{=} \dot{\mathrm{H}}_D(B)_{(z)}).$$

An Important Special Case

- If $\bigoplus$ is preserved by $(\div)$ (resp. $(.)$) then

$$\mathrm{T}_D(A)_{(z)} = \coprod_{x\in G} A_{(x)}.D_{(-x+z)} \stackrel{\text{def}}{=} \dot{\mathrm{T}}_D(A)_{(z)}$$

$$(\textit{resp.}\;\; \mathrm{H}_D(B)_{(z)} = \prod_{x\in G} B_{(x)} \div D_{(-z+x)} \stackrel{\text{def}}{=} \dot{\mathrm{H}}_D(B)_{(z)}).$$

- **Theorem**(D.& Hashemi 2000)
  For any $D \in \Phi$ we have $\dot{\mathrm{T}}_D \dashv \dot{\mathrm{H}}_D$ (i.e. $\dot{\mathrm{T}}_D$ is the left adjoint of $\dot{\mathrm{H}}_D$).

AN IMPORTANT SPECIAL CASE

- If $\bigoplus$ is preserved by $(\div)$ (resp. $(.)$) then

$$\mathrm{T}_D(A)_{(z)} = \coprod_{x \in G} A_{(x)}.D_{(-x+z)} \overset{\text{def}}{=} \dot{\mathrm{T}}_D(A)_{(z)}$$

$$(\textit{resp.}\ \ \mathrm{H}_D(B)_{(z)} = \prod_{x \in G} B_{(x)} \div D_{(-z+x)} \overset{\text{def}}{=} \dot{\mathrm{H}}_D(B)_{(z)}).$$

- **Theorem**(D.& Hashemi 2000)
For any $D \in \Phi$ we have $\dot{\mathrm{T}}_D \dashv \dot{\mathrm{H}}_D$ (i.e. $\dot{\mathrm{T}}_D$ is the left adjoint of $\dot{\mathrm{H}}_D$).
- **Theorem**(D.& Hashemi 2000)
There exist natural isomorphisms $\tilde{a}, \tilde{l}, \tilde{r}$ and $\tilde{c}$ such that for any $D \in \mathrm{obj}(\Phi)$, $\dot{\Phi} = (\Phi, \dot{\mathrm{T}}_D, \dot{\mathrm{H}}_D, P^{0,I}, \tilde{a}, \tilde{l}, \tilde{r}, \tilde{c})$ is a symmetric closed monoidal category.

- A point $P^{g,V} \in \Phi$ with value $V \in \mathrm{obj}(\mathcal{V}_0)$ at coordinate $g \in G$ is defined (up to isomorphism) as

$$P^{g,V}(x) \stackrel{\text{def}}{=} \begin{cases} V & x = g \\ -\infty & x \neq g. \end{cases}$$

General Reconstruction Theorem

- A point $P^{g,V} \in \Phi$ with value $V \in \mathrm{obj}(\mathcal{V}_0)$ at coordinate $g \in G$ is defined (up to isomorphism) as

$$P^{g,V}(x) \stackrel{\text{def}}{=} \begin{cases} V & x = g \\ -\infty & x \neq g. \end{cases}$$

- For any $\dot{\Phi}$–functor F : $\dot{\Phi} \longrightarrow \dot{\Phi}$, Ker(F) is defined to be the category of elements of the Set–functor $[P^{0,I}, F-]_{\Phi}$.

General Reconstruction Theorem

- A point $P^{g,V} \in \Phi$ with value $V \in \mathrm{obj}(\mathcal{V}_0)$ at coordinate $g \in G$ is defined (up to isomorphism) as

$$P^{g,V}(x) \stackrel{\text{def}}{=} \begin{cases} V & x = g \\ -\infty & x \neq g. \end{cases}$$

- For any $\dot{\Phi}$–functor $\mathrm{F} : \dot{\Phi} \longrightarrow \dot{\Phi}$, $Ker(\mathrm{F})$ is defined to be the category of elements of the Set–functor $[P^{0,I}, F-]_{\Phi}$.
- **Theorem**(D.& Hashemi 2000)
  Let $\mathrm{F} : \dot{\Phi} \longrightarrow \dot{\Phi}$ be a $\dot{\Phi}$–functor such that F preserves the internal Hom of $\dot{\Phi}$, $\dot{\mathrm{H}}_D$, for any $D \in \Phi$; and $Ker(\mathrm{F})$ is a $\mathcal{D}$–type diagram–scheme. Then F has a representation as a $\mathcal{D}$–colimit of representables as

$$\mathrm{F}(A) \simeq \mathrm{Colim}_{(D,d) \in Ker(F)} \dot{\mathrm{H}}_D(A).$$

Example I

- Consider $G = (\mathbf{R}, +)$ and $\mathcal{V} = (\mathbf{R}, +, \leq)$ with two universal bounds $+\infty$ and $-\infty$. Then $\dot{\mathrm{T}}$ is the Minkowski addition and $\dot{\mathrm{H}}$ is the erosion operator.

Example I

- Consider $G = (\mathbf{R}, +)$ and $\mathcal{V} = (\mathbf{R}, +, \leq)$ with two universal bounds $+\infty$ and $-\infty$. Then $\dot{\mathrm{T}}$ is the Minkowski addition and $\dot{\mathrm{H}}$ is the erosion operator.
- Also, it can be shown that in this case being a $\dot{\Phi}$–functor is equivalent to the definition of a translation invariant operator, where we have

$$Ker(\mathrm{F}) = \{A \in \dot{\Phi} \mid \mathrm{F}(A)(0) \geq 0\}.$$

Example I

- Consider $G = (\mathbf{R}, +)$ and $\mathcal{V} = (\mathbf{R}, +, \leq)$ with two universal bounds $+\infty$ and $-\infty$. Then $\dot{\mathrm{T}}$ is the Minkowski addition and $\dot{\mathrm{H}}$ is the erosion operator.
- Also, it can be shown that in this case being a $\dot{\Phi}$–functor is equivalent to the definition of a translation invariant operator, where we have

$$Ker(\mathrm{F}) = \{A \in \dot{\Phi} \mid \mathrm{F}(A)(0) \geq 0\}.$$

- Note that in this case the general reconstruction theorem yields the classical reconstruction theorem for TI operators.

- Now consider the following maps for a fixed $D \in \Phi$,

  $\tilde{\mathrm{T}}_D : \Phi \longrightarrow \mathcal{V}$,

  $$\tilde{\mathrm{T}}_D(A) = \prod_{|J|<\infty} (\bigoplus_{j\in J} W_j)^{[A_{(x)}, \oplus_j(\Delta_{W_j} \div D_{(-x+z_j)})]_\Phi},$$

  $\tilde{\mathrm{H}}_D : \Phi \longrightarrow \mathcal{V}$,

  $$\tilde{\mathrm{H}}_D(B) = \coprod_{|J|<\infty}^{[\oplus_j(\Delta_{V_j} \cdot D_{(-z_j+x)}), B_{(x)}]_\Phi} (\bigoplus_{j\in J} V_j),$$

  where $\bigoplus$ is a suitable associative bifunctor.

A SIMPLE CASE

- As a simple case we have,

$$\ddot{\mathrm{T}}_D(A) = \prod_j W_j^{[A_{(x)}, \Delta_{W_j} \div D_{(-x+z_j)}]_\Phi} = \prod_z \dot{\mathrm{T}}_D(A)_{(z)},$$

and

$$\ddot{\mathrm{H}}_D(B) = \coprod_j {}^{[\Delta_{V_j} \cdot D_{(-z_j+x)}, B_{(x)}]_\Phi} V_j = \coprod_z \dot{\mathrm{H}}_D(A)_{(z)}.$$

A SIMPLE CASE

- As a simple case we have,

$$\ddot{\mathrm{T}}_D(A) = \prod_j W_j^{[A_{(x)}, \Delta_{W_j} \div D_{(-x+z_j)}]_\Phi} = \prod_z \dot{\mathrm{T}}_D(A)_{(z)},$$

and

$$\ddot{\mathrm{H}}_D(B) = \coprod_j {}^{[\Delta_{V_j} \cdot D_{(-z_j+x)}, B_{(x)}]_\Phi} V_j = \coprod_z \dot{\mathrm{H}}_D(A)_{(z)}.$$

- **Theorem**(D.& Hashemi 2000)
If the tensor of $\mathcal{V}$ is preserved by coproducts, then there exist a natural composition law and an identity map such that $[D, B] \stackrel{\text{def}}{=} \ddot{\mathrm{H}}_D(B)$, as the internal Hom, turns Φ into a $\mathcal{V}$–category.

A SIMPLE CASE

- As a simple case we have,

$$\ddot{\mathrm{T}}_D(A) = \prod_j W_j^{[A_{(x)}, \Delta_{W_j} \div D_{(-x+z_j)}]_\Phi} = \prod_z \dot{\mathrm{T}}_D(A)_{(z)},$$

and

$$\ddot{\mathrm{H}}_D(B) = \coprod_j {}^{[\Delta_{V_j} \cdot D_{(-z_j+x)}, B_{(x)}]_\Phi} V_j = \coprod_z \dot{\mathrm{H}}_D(A)_{(z)}.$$

- **Theorem**(D.& Hashemi 2000)
  If the tensor of $\mathcal{V}$ is preserved by coproducts, then there exist a natural composition law and an identity map such that $[D, B] \stackrel{\text{def}}{=} \ddot{\mathrm{H}}_D(B)$, as the internal Hom, turns Φ into a $\mathcal{V}$–category.
- Can you prove a similar theorem for the general case?

Example II

- Let $G = (\mathbf{R}, +)$ and $\mathcal{V} = (\mathbf{R}^+, ., \geq)$. Then

$$\ddot{\mathrm{H}}_D(B) = \inf_z \left(\sup_x \left(B_{(x)} \div D_{(-z+x)}\right)\right)$$

in the ordinary order of real numbers.

Example II

- Let $G = (\mathbf{R}, +)$ and $\mathcal{V} = (\mathbf{R}^+, ., \geq)$. Then

$$\ddot{\mathrm{H}}_D(B) = \inf_z \left(\sup_x \left(B_{(x)} \div D_{(-z+x)}\right)\right)$$

  in the ordinary order of real numbers.
- Let $\oplus = +$ be the ordinary addition of real numbers. Then,

$$\tilde{\mathrm{H}}_D(B) = \inf_{c_j, z_j} \{\Sigma_j \; c_j \mid \;\; \forall\, x \;\; B_{(x)} \leq \Sigma_j \; c_j D_{(-z_j+x)}\} = (B : D)$$

Example II

- Let $G = (\mathbf{R}, +)$ and $\mathcal{V} = (\mathbf{R}^+, ., \geq)$. Then

$$\ddot{\mathrm{H}}_D(B) = \inf_z \left(\sup_x \left(B_{(x)} \div D_{(-z+x)}\right)\right)$$

in the ordinary order of real numbers.

- Let $\oplus = +$ be the ordinary addition of real numbers. Then,

$$\tilde{\mathrm{H}}_D(B) = \inf_{c_j, z_j} \{\Sigma_j\ c_j \mid\ \forall\ x\ \ B_{(x)} \leq \Sigma_j\ c_j D_{(-z_j+x)}\} = (B : D)$$

- This is the Haar fraction!

Some after-thoughts

- In the general setting a closed monoidal category with suitable completeness properties is a suitable model for the valuation space that generalizes both functional and fuzzy approaches.

Some after-thoughts

- In the general setting a closed monoidal category with suitable completeness properties is a suitable model for the valuation space that generalizes both functional and fuzzy approaches.
- The fuzzy set approach introduces applications based on Boolean functions that give rise to non-linear operators.

Some after-thoughts

- In the general setting a closed monoidal category with suitable completeness properties is a suitable model for the valuation space that generalizes both functional and fuzzy approaches.
- The fuzzy set approach introduces applications based on Boolean functions that give rise to non-linear operators.
- Can one define a limit process that simulates integration in the general setting?

Some after-thoughts

- In the general setting a closed monoidal category with suitable completeness properties is a suitable model for the valuation space that generalizes both functional and fuzzy approaches.
- The fuzzy set approach introduces applications based on Boolean functions that give rise to non-linear operators.
- Can one define a limit process that simulates integration in the general setting?
- Can one develop transformation techniques that facilitate a design process based on some wanted behaviour of operators.
  (This should be definitely related to some new understandings and dimensions in system design.)

A brief history of fuzzy sets

- J. Łukasiewicz (1920), Gödel (1932): Many-valued logic.
- M. Black (1937): Vagueness
- H. Weyl (1946): Calculus of vague predicates.
- H. Reichenbach (1949): Probability logic.
- L. A. Zadeh (1965-1978): Possibility theory.
- J. A. Goguen (1967): Extension to lattice-valued maps.

note

These contributions led to what is known as fuzzy logic, with vast applications in decision making, approximate reasoning, control theory, information processing, ...

Foundations of fuzzy set-logic theory

- D. Klaua (1965): A hierarchy of fuzzy sets.
- J. A. Goguen (1967): The category of L-fuzzy sets.
- D. S. Scott (1971): Continuous lattice theory.
- E. W. Chapin Jr. (1974): First axiomatization.
- J. Lake (1976): Fuzzy sets as multisets.
- M. Eytan (1981): A topos-logical point of view.
- U. Höhle, L. N. Stout: Foundation and Monoidal approach.
- O. Wyler (1991): Quasitopoi.

Fuzzy set theory and Mathematics of 21'st century

Question

Where does fuzzy set theory stand in modern mathematics?

Some motivations:

- A well-developed mathematical discipline must have its own formalism (almost developed).
- A well-developed mathematical discipline must have its own main problems (is this developed enough?).
- A well-developed mathematical discipline must have its own connections to the other mathematical disciplines.

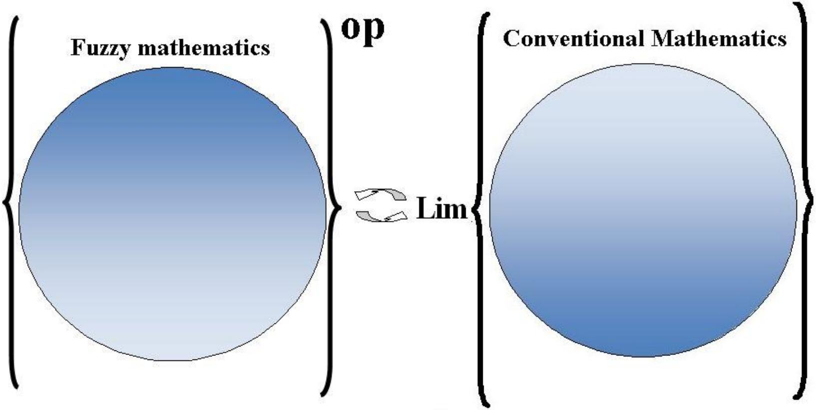
Fuzzy set theory and Mathematics of 21'st century

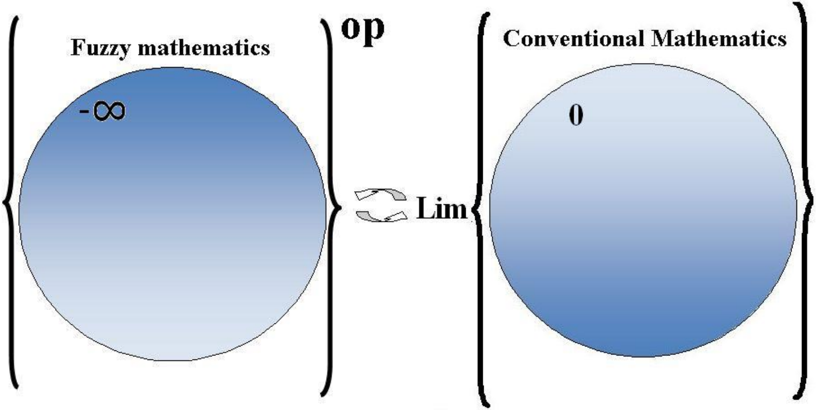
Personal opinion!

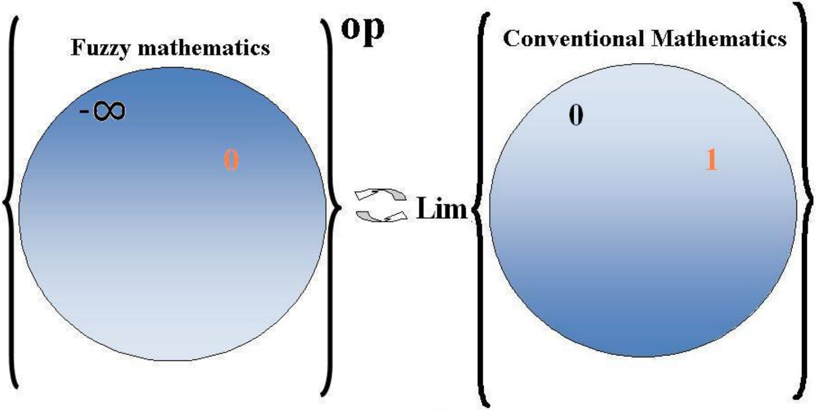
I believe that these connections are much deeper than what we usually present, and that the position of fuzzy set theory is quite central in mathematics of 21'st century.

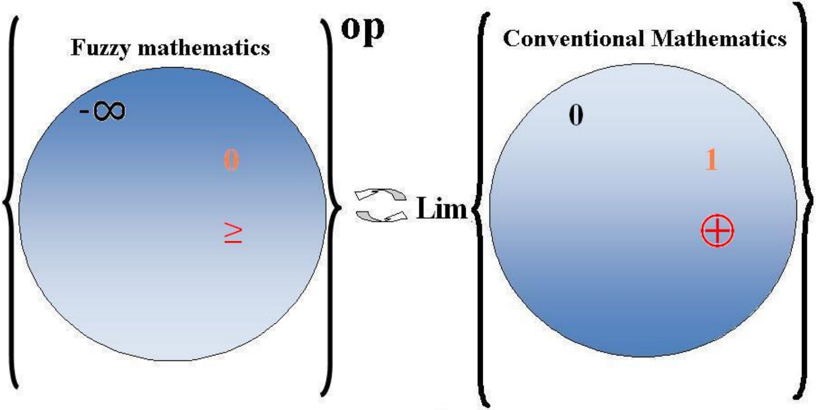
Some background and motivations are:

- Theory of Dioids (M. Akian, G. Cohen, S. Gaubert, M. Gondran, M. Minoux, J. P. Qyadrat, V. P. Maslov, ...)
- Toll sets (J. P. Aubin, O. Dordan, AD, ...).
- Sheaf-theoretic approach (U. Höhle, AD, ...)
- Mathematical system theory (L. A. Zade, AD, ...).

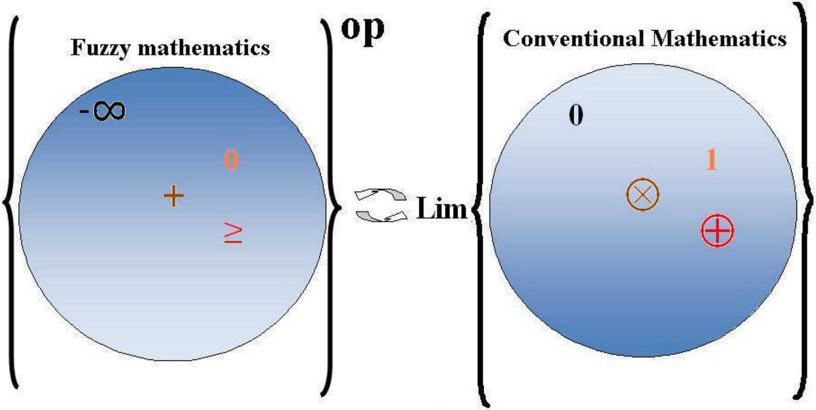

Fuzzy mathematics
op
Lim
Conventional Mathematics

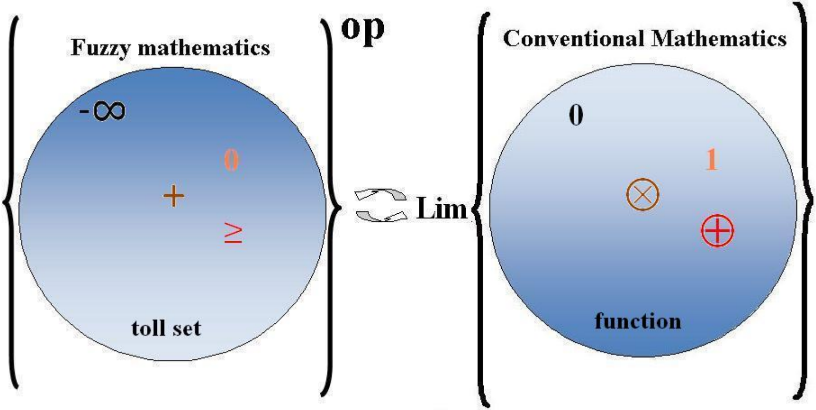

op
Fuzzy mathematics
-∞
Lim
Conventional Mathematics
0

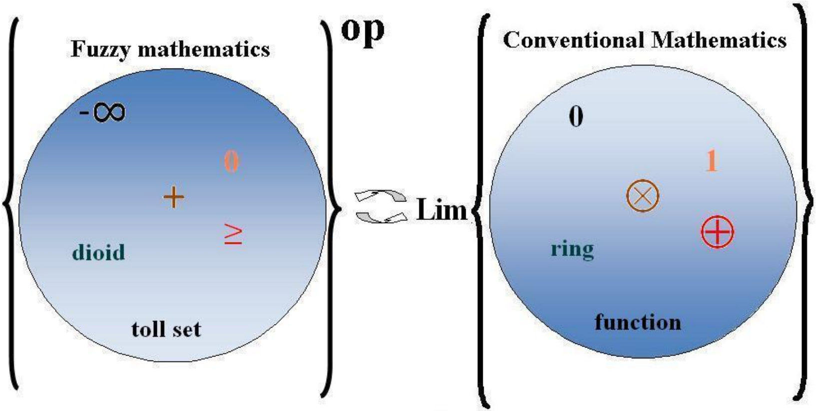

op
Fuzzy mathematics
-∞
0
Lim
Conventional Mathematics
0
1



Fuzzy mathematics
-∞
0
≥
op
Lim
Conventional Mathematics
0
1
⊕


Fuzzy mathematics
-∞
0
+
≥
op
Lim
Conventional Mathematics
0
1
⊗
⊕


op
Fuzzy mathematics
-∞
0
+
≥
toll set
Lim
Conventional Mathematics
0
1
⊗
⊕
function



Fuzzy mathematics
op
-∞
0
+
≥
dioid
toll set
Lim
Conventional Mathematics
0
1
⊗
⊕
ring
function

Definition of a semiring

- A **semiring** $(D, \oplus, \otimes)$ is an algebraic structure which satisfies the following properties:
  1. $\oplus$ and $\otimes$ are associative binary operations on D,
  2. $\oplus$ is commutative and have an identity element ϵ,
  3. $\otimes$ have an identity element e. Moreover it is distributive with respect to $\oplus$
  4. for all $d \in D$ we have $d \otimes \epsilon = \epsilon \otimes d = \epsilon$.

Subclasses of semirings

The class of semirings can be naturally subdivided into two disjoint sub-classes depending on whether the operation $\oplus$ satisfies one of the following two properties:

1. The operation $\oplus$ endows the set D with a group structure;
2. The operation $\oplus$ endows the set D with a canonically ordered monoid structure.

note

(1) and (2) cannot be satisfied simultaneously. In case (1), we are led to the well-known Ring structure, and in case (2) we are lead to the Dioid structure.

Examples of dioids

- $(\mathbb{Z}, +, \times)$ is a ring but not a dioid.
- $(\mathbb{N}, +, \times)$ is a dioid which is not idempotent.
- $(\mathbb{N}, lcm, gcd)$ is a doubly idempotent dioid.
- $\mathbb{R}_{max} \stackrel{\text{def}}{=} (\mathbb{R} \cup \{-\infty\}, max, +)$ is an idempotent dioid which is called the max-plus algebra.
- $(\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}, max, min)$ is an idempotent dioid.
- $(2^{\mathbb{R}^2}, \cup, +)$ is a dioid, where for any $A, B \subseteq \mathbb{R}^2$

$$A + B \stackrel{\text{def}}{=} \{y + z | y \in A, \ \& \ z \in B\}.$$

- $(Reg(\Sigma), \cup, concat)$ is a dioid.

Definition of a dioid

- A **Dioid** is a triple $(D, \vee, \otimes)$ that satisfies the following conditions:
  1. $(D, \vee)$ is a commutative monoid with neutral element ϵ.
  2. $(D, \otimes)$ is a monoid with neutral element e.
  3. for all $d \in D$ we have $d \otimes \epsilon = \epsilon \otimes d = \epsilon$.
  4. $\otimes$ is (right and left) distributive with respect to $\vee$.
  5. $\vee$ induces an order define as:
     $$a \leq b \ \Leftrightarrow \ \exists\, c, \ \ b = a \vee c.$$

note

A dioid is a semiring $(D, \vee, \otimes)$ in which the operation $\vee$ induces a natural order structure.

Idempotent dioids

note

In general in a dioid $(D, \vee, \otimes)$ the canonical order is compatible with the operations $\vee$ and $\otimes$.

A dioid is **idempotent** iff $\forall\ a \in D,\ a \vee a = a$.
Note that this implies the following fact,

$$a \leq b \ \Leftrightarrow\ \exists\ c,\ \ b = a \vee c \ \Leftrightarrow\ a = a \vee b.$$

note

Sometimes this condition is added to the definition of a dioid as an axiom.

Matrix multiplication in $\mathbb{R}_{max}$

Example

For matrix $A = \begin{pmatrix} 3 & 7 \\ 2 & 4 \end{pmatrix}$ in $Mat_{n \times n}(\mathbb{R}_{max})$ we have

$$A^2 = \begin{pmatrix} max(3+3, 7+2) & max(3+7, 7+4) \\ max(2+3, 4+2) & max(2+7, 4+4) \end{pmatrix} = \begin{pmatrix} 9 & 11 \\ 6 & 9 \end{pmatrix}$$

in linear algebra we have

$$A^2 = \begin{pmatrix} +(3 \times 3, 7 \times 2) & +(3 \times 7, 7 \times 4) \\ +(2 \times 3, 4 \times 2) & +(2 \times 7, 4 \times 4) \end{pmatrix} = \begin{pmatrix} 23 & 49 \\ 14 & 30 \end{pmatrix}$$

Dioids appear in:

- Algebra (Ordered Monoids, Complete Closed Monoidal Categories, Lattice Theory)
- Matrix Algebras (Nonlinear Perron-Ferobenius Theory, Bideterminants, Generalized Cayley-Hamilton Theorems, semi-modules, Moduloids)
- Algebraic Geometry (Tropical Algebras)
- Analysis, System Theory and Mathematical Physics (HJB and Bürgers Equations, Generalized Integration, Nonlinear Systems, Min-Max Analysis)

Dioids appear in:

- Combinatorics (Graph Theory)
- Probability (Markov Chains, Markov Decision Processes, Generalized Capacities)
- Optimization/OR (Dynamic Programming, Shortest Path Problems)
- Computer Science (Network Problems, Fuzzy systems, Discrete Dynamics)

Dequantization of $\mathbb{R}^+$

Let $r \in \mathbb{R}^+$ be a positive number and consider the following transform for the positive (Planck) constant h:

$$\sim: \mathbb{R}^+ \to \mathbb{R}_h \quad r \mapsto \tilde{r} = h \ln r.$$

Now, define

$$r \vee s \stackrel{\text{def}}{=} h \ln(e^{\frac{r}{h}} + e^{\frac{s}{h}}) \quad \text{and} \quad r \otimes s \stackrel{\text{def}}{=} h \ln(e^{\frac{r}{h}} \times e^{\frac{s}{h}}) = r + s.$$

note

This transformation preserves the properties of the algebraic operations between $\mathbb{R}^+$ and its image $\mathbb{R}_h \stackrel{\text{def}}{=} \widetilde{\mathbb{R}^+}$.

Dequantization of $\mathbb{R}^+$

Consider $\mathbb{R}_h$ and let h tends to zero. Then it is not hard to verify that in the limit, $\mathbb{R}_{0^+}$ is isomorphic to the max-plus algebra $\mathbb{R}_{max} \stackrel{\text{def}}{=} (\mathbb{R} \cup \{-\infty\}, max, +)$. Hence, the max-plus algebra $\mathbb{R}_{max}$ can be thought as a deformation of $\mathbb{R}^+$, and $\mathbb{R}^+$ can be thought of as a quantized version of $\mathbb{R}_{max}$. Note that in this setting,

$$\tilde{1} = 0 \quad \text{and} \quad \tilde{0} = -\infty.$$

note

This is an example of the correspondence principle borrowed from Quantum Mechanics.

Dequantization of the heat equation

Consider the heat equation and apply the transform
$r \mapsto w = -h \ln u$,

$$\frac{\partial u}{\partial t} = \frac{h}{2}\frac{\partial^2 u}{\partial x^2} \quad \mapsto \quad \frac{\partial w}{\partial t} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - \frac{h}{2}\frac{\partial^2 w}{\partial x^2} = 0,$$

and pass to the limit $h \to 0$ to get,

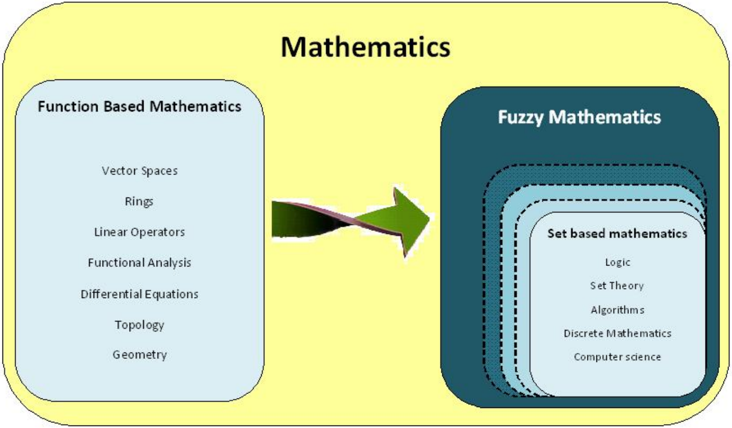
$$\mapsto \quad \frac{\partial w}{\partial t} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 = 0,$$

which is (a special case) of the Hamilton-Jacobi-Bellman equation, with linear properties for solutions in $\mathbb{R}_{min}$.

Mysterious duality

Observe the duality between the universe of functions and the universe of sets!

- We work with sets and functions in conventional mathematics.



What is Fuzzy Mathematics?

Semirings and graphs

Let S be a semiring and $A \in Mat_{n \times n}(S)$ be a matrix with entries in S. Then one may naturally identify A with a weighted directed graph structure, on n vertices, where there is a directed edge from the vertex u to the vertex v of weight $\epsilon \neq a_{uv} \in S$ if and only if this value is the entry in the row u and the column v of the matrix A.

note

Standard notions as paths, cycles and their weights are defined naturally.

The shortest path problem

definition

For any matrix A in $(D, \oplus, \otimes)$ we define A^* as follow

$$A^* \stackrel{\text{def}}{=} I \oplus A \oplus A^2 \oplus \cdots \oplus A^k \oplus \cdots$$

Questions

When does this limit A^* exists?
How can we compute it?

The shortest path problem

Let $S = \mathbb{R}_{min} \overset{\text{def}}{=} (\mathbb{R} \cup \{+\infty\}, \wedge, +)$ and $A \in Mat_{n \times n}(S)$ be the adjacency matrix of the weighted directed graph G on the vertex set $\{v_1, v_2, \ldots, v_n\}$. Then the shortest path problem is equal to find the matrix $D = [d_{ij}]$ such that d_{ij} is the minimum of the weights of all paths starting from v_i and ending at v_j. Note that in this setting we have,

$$D = A^* = I \wedge A \wedge A^2 \wedge \cdots \wedge A^k \wedge \cdots$$

Questions

When does this limit A^* exists?
How can we compute it?

A simple optimization problem

Let $S = \mathbb{R}_{max} \stackrel{\text{def}}{=} (\mathbb{R} \cup \{-\infty\}, \vee, +)$ and $A \in Mat_{n \times n}(S)$ be the adjacency matrix of the weighted directed graph G on the vertex set $\{v_1, v_2, \ldots, v_n\}$. Interpret a_{ij} as the profit of going from v_i to v_j and let $f_i \in \mathbb{R}$ be a terminal prize of ending at vertex v_i. Therefore, the solution of maximizing the income after k steps is equal to $D_k = A^k f$. Also, an overall maximizing solution is

$$D = A^* f.$$

Questions

When does this limit $A^* f$ exists?
How can we compute it?

Inf-convolution of quadratic forms

For any $m \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$ define

$$Q_{m,\sigma}(x) \stackrel{\text{def}}{=} \frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2 \quad \text{for} \ \ \sigma \neq 0,$$

$$Q_{m,0}(x) \stackrel{\text{def}}{=} \delta_m(x) \stackrel{\text{def}}{=} \begin{cases} 0 & x = m \\ +\infty & \text{otherwise.} \end{cases}$$

Also, consider the inf-convolution $f \otimes g$ as $f \otimes g(z) \mapsto \inf_{x+y=z} f(x) + g(y)$. Then,

$$(Q_{m_1,\sigma_1} \otimes Q_{m_2,\sigma_2})(x) = Q_{m_1+m_2,\sqrt{\sigma_1+\sigma_2}}(x).$$

Dynamic programming I

Consider a very simple discrete decision (control) process,

$$x(0)\ \text{Given}, \quad x(n+1) = x(n) - u(n),$$

along with the cost function

$$Cost(N) = \min_{u(0),u(1),\ldots,u(N-1)} \left(\varphi(x(N)) + \sum_{i=0}^{N-1} c(u(i)) \right).$$

note

Functions c and φ satisfy typical conditions as being convex, lower-semicontinuous, and zero at their minimum, etc.

Dynamic programming II

Define

$$v(n,x) \stackrel{\text{def}}{=} \min_{u(n),u(1),\ldots,u(N-1)} \left(\varphi(x(N)) + \sum_{i=n}^{N-1} c(u(i)) \mid x(n) = x \right),$$

and note that it satisfies the dynamic programming equation

$$v(n,x) = \min_u \left(c(u) + v(n+1, x-u) \right), \quad v(N,x) = \varphi(x).$$

This equation can be written as follows using inf-convolution $\otimes$,

$$v(n,.) = c \otimes v(n+1,.), \quad v(N,.) = \varphi.$$

note

Hence, the solution is $v(0,.) = c^N \otimes \varphi$.

The Brownian decision process I

Again consider the discrete time decision (control) process,

$$x(0) \text{ Given}, \quad x(t+h) = x(t) - u(t),$$

along with the cost function

$$\min_u \left(\Phi(x(T)) + \sum_{i=0}^{T/h-1} \frac{u(ih)^2}{2h} \right).$$

This can also be solved through the dynamic programming equation

$$v(t,x) = \min_u \left(\frac{u^2}{2h} + v(t+h, x-u) \right), \quad v(T,.) = \Phi.$$

The Brownian decision process II

Use the change of control $u = hw$ in the dynamic programming equation and let $h \mapsto 0$ to get

$$\frac{\partial v}{\partial t} + \min_w \left(-w \frac{\partial v}{\partial x} + \frac{w^2}{2} \right) = 0, \quad v(T, .) = \Phi.$$

i.e. equivalent to the Hamilton-Jacobi-Bellman equation,

$$\frac{\partial v}{\partial t} - \frac{1}{2} \left(\frac{\partial v}{\partial x} \right)^2 = 0, \quad v(T, .) = \Phi.$$

note

In this sense the theory of Brownian decision processes can be thought as the dequantization of the theory of heat equations.

Simple equations in $\mathbb{R}_{max}$

Let $\mathbb{R}_{max} \stackrel{\text{def}}{=} (\mathbb{R} \cup \{-\infty\}, \vee, +)$ and $A \in Mat_{n \times n}(\mathbb{R}_{max})$ be the adjacency matrix of the weighted directed graph G. Then, If there are only circuits of nonpositive weight in G, there is a solution to $x = Ax \vee b$ which is given by $x = A^* b$. Moreover, if the circuit weights are negative, the solution is unique.

note

Hint: $A(A^* b) \vee b = (e \vee AA^*)b = A^* b$.

Simple discrete dynamics in $\mathbb{R}_{max}$

Let $A \in Mat_{n\times n}(\mathbb{R}_{max})$ and consider the dynamics

$$x(n+1) = Ax(n), \quad x(0) \text{ given},$$

whose solution is $x(n) = A^n x(0)$. Naturally, part of the analysis depend on the existence of eigenvalues and eigenfunctions for A, i.e. the existence of λ and f such that $Af = \lambda f$.

Eigenvalues and eigenvectors in $\mathbb{R}_{max}$

If $A \in Mat_{n\times n}(\mathbb{R}_{max})$ is irreducible, or equivalently if the corresponding graph G is strongly connected, there exists one and only one eigenvalue (but possibly several eigenvectors). This eigenvalue is equal to the maximum cycle mean of the graph G, i.e.

$$\lambda = \max_{c} \frac{|c|_w}{|c|_1},$$

where c ranges over cycles of the graph G.

Eigenvalues and eigenvectors (examples)

Example

Eigenfunctions are not unique.

$$\begin{pmatrix} 1 & e \\ e & 1 \end{pmatrix} \begin{pmatrix} e \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ e \end{pmatrix} {=}1 \begin{pmatrix} e \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & e \\ e & 1 \end{pmatrix} \begin{pmatrix} -1 \\ e \end{pmatrix} = \begin{pmatrix} e \\ 1 \end{pmatrix} {=}1 \begin{pmatrix} -1 \\ e \end{pmatrix}$$

Eigenvalues and eigenvectors (examples)

Example

Eigenvalues are not unique.

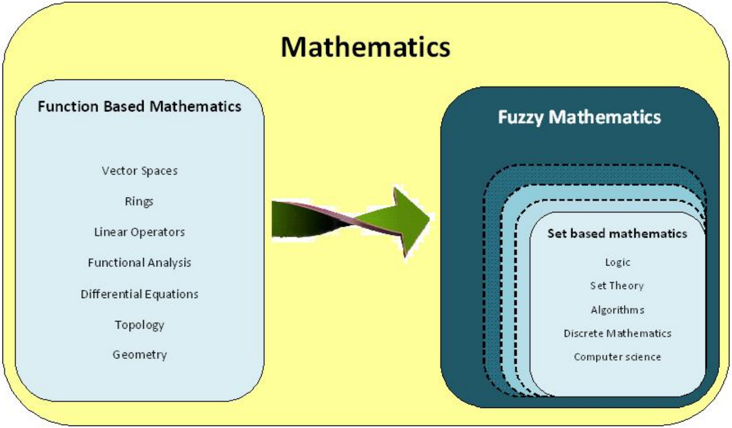
$$\begin{pmatrix} 1 & \epsilon \\ \epsilon & 2 \end{pmatrix} \begin{pmatrix} e \\ \epsilon \end{pmatrix} = 1 \begin{pmatrix} e \\ \epsilon \end{pmatrix}$$

$$\begin{pmatrix} 1 & \epsilon \\ \epsilon & 2 \end{pmatrix} \begin{pmatrix} \epsilon \\ e \end{pmatrix} = 2 \begin{pmatrix} \epsilon \\ e \end{pmatrix}$$

Results of linear algebra in $\mathbb{R}_{max}$

A large number of results of conventional algebra can be extended to $\mathbb{R}_{max}$, and even to matrices over an arbitrary semiring, as

- The Cayley-Hamilton theorem.
- Determinants
- Polynomial functions
- Formal series



What is Fuzzy Mathematics?
Can we call it a dioid-based mathematics?

Epilogue: the mysterious and intriguing duality

- Measure theory stands in between of the two sides of mathematics we discussed.
- The natural formalism for this study is the theory of enriched sheaves.
- A well-defined theory for this reveals the essence of the duality we talked about.
- This theory is also central to many other fundamental problems of modern mathematics.
- An introduction to this subject needs a new talk!

Personal opinion!!

Lets think about a theory of enriched sheaves!!!!!

Thank you for your attention.