Theory of Computation

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Outline

Theory of computation: Origins (computability)

Al-Kharazmi
(∼ 780-850 AD)

Kurt Gödel
(1906 - 1978)

The fundamental question in early days

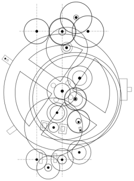
Can we provide a computational solution to any problem?

Some fundamental questions to ask

- Is it possible to precisely define the concepts "computation" and "algorithm"?
- Does any given problem have a constructive "algorithmic solution"?
- Do natural hard problems exist? what are the consequences of answers YES or NO to this question?
- Is it possible to find the best constructive algorithmic solution to a given problem?

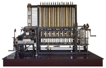
Note the practical importance as well as
the theoretical nature of these problems!
The history and results are fascinating and surprising!
Let's talk about it.

Prehistory < 1900



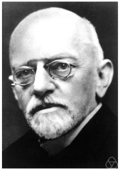
- Abacus (~ 3000 B.C.): Widely used for daily computations (even some evidence of use in Babylonia (present-day Iraq) around ~ 3000 B.C.).
- Elements of Euclid (~ 300 B.C.): The first rigorous axiomatization process in mathematics (geometry) and the birthday of the concept of a rigorous "proof".
- Antikythera mechanism (~ 80 B.C.): Discovered in 1901, within an ancient Greek shipwreck off the island of Antikythera.
- *Muhammad ibn Mūsā al-Khwārizmī* (~ 800 C.E.): Birthday of the concepts "algebra" and "algorithm".

Prehistory < 1900



- Mechanical adding machine ($\sim 1620 - 1640$): Wilhelm Schickard, Blaise Pascal, Gottfried Wilhelm Leibniz.
- Difference Engine ($1791 - 1871$): Charles Babbage .
- First program ($1815 - 1852$): Ada Augusta Byron, Countess of Lovelace.
- Gottlob Frege ($1848 - 1925$): Birthday of modern logic.
- Giuseppe Peano ($1858 - 1932$): Standard axiomatization of the natural numbers and birthday of recursive definitions.

A mathematical boost 1900 – 1940



Kurt Gödel

Alan Turing

- David Hilbert (1900): addressed the International Congress of Mathematicians with three main questions on computability.
- Kurt Gödel (1930): Answered two important questions on consistency and completeness.
- Alan Turing (1936): Introduced a formal model of a computer, the *Turing machine*, and the *halting problem*.
- Contributions of Church, Turing, Post, Kleene, ... on the concept of computation and recursion theory motivated by the concept of a proof of a true mathematical statement.

Digital computers 1940 – 1950

- Z3 (1941): the first operational, general-purpose, program-controlled calculator built by Konrad Zuse.
- Colossus (1943): Built by British to help Alan Turing breaking the code behind the German machine, the Enigma.
- Mark I electromechanical computer (1944): The calculations required for ballistics during World War II led to this construction by Howard H. Aiken.
- EDVAC (1944): Mauchly, Eckert, and John von Neumann.
- ENIAC (1946): Built at the Moore School at the University of Pennsylvania.
- Invention of the transistor (1947): By John Bardeen, Walter Brattain, and William Shockley.
- Invention of magnetic core memory ($\sim$ 1949): By Jay Forrester.

Compiler design 1950 – 1960

- Invention of the notion of a compiler (1951): By Grace Murray Hopper at Remington Rand.
- First FORTRAN compiler (1957): John Backus and others.
- LISP and ALGOL (1958): John McCarthy and Alan Perlis, John Backus, Peter Naur and others.
- Integrated circuits (1959): Jack Kilby (Texas Instruments) and Robert Noyce (Fairchild Semiconductor).

The CS discipline $1960 - 1970$

- The rise of automata theory and the theory of formal languages $(1960's)$: Noam Chomsky, Michael Rabin and others.
- Computer science as a discipline (1962): The first computer science department was formed at Purdue University.

Theory of computation: A culmination of ideas coming from digital design, compiler design, recursion theory and complexity.

Birthday of a scientific discipline

- NP-completeness (1971): Stephen Cook's seminal paper.
- NP-completeness (1973): Leonid Levin's article.
- Design of CRAY-1 (1976): Seymour Cray.

Modern Theory of computation: Theoretical foundations for digital circuit design, Compiler design, computability and analysis of algorithms.

Theory of computation: Origins (old times up to 1936)

- Numeric computations (3000 B.C. - Now): day-to-day life, science and technology.
- Geometry (3000 B.C. - Now): As a calculator before Descartes and as one of the best axiomatized mathematical theories afterwards.
- Hilbert (1900): Can every true statement be proven automatically (in a finitely axiomatized system)? (Note: at the time no notion of a modern computer was available!)
- Gödel's incompleteness theorem (1931): In a finitely axiomatized system which is strong enough to express the arithmetic of natural numbers, some true statements are unprovable!
- Turing's undecidability theorem (1936): Undecidable statements (yes/no problems) do exist!
- Church–Turing thesis (1936): A quest for a correct definition for an “algorithm".

Theory of computation: motivations (modern times)

- (1940's): Design and verification of digital systems.
- (1950's): Design and verification of software systems.
- (1960's): Design and verification of efficient numerical algorithms.
- Stephen Cook (1971) also Leonid Levin (1973): Existence of natural NP-complete problems.
- (1980's): Analysis of yes/no problems through the theory of formal languages.
- (1990's): Theory of computational complexity and design of efficient algorithms, and inapproximability.

Theory of computation: the coding trick

A problemtype P consists of the following data:

- Constants: ....
- Given input: .....
- Query: ....?

An example

- Constants: 3.
- Given input: the integer n.
- Query: Is n divisible by 3?

Main question: How do you provide the data?

The membership problemtype

- Constants: Σ a finite set of alphabets and a subset $L \subseteq \Sigma^*$.
- Given input: a word $x \in \Sigma^*$.
- Query: Is it true that $x \in L$?

Fact: Any yes-no problem as P can be reduced to a membership problem for some subset (i.e. a language) L_P.

Main question: How do you provide the data?

A computational model (machine)

This leads to an interpretation of ToC as one of the variants to provide finite presentations for infinite sets!

Such a finite presentation is a program or more abstractly a computing machine consisting of:

- A hardware (i.e. a control unit).
- A memory.
- A mechanism to read the input.
- A mechanism to write the output.

working as a discrete dynamical system satisfying the local property.

Theory of computation: early impacts

There was a controversy regarding the definition of computation during the early days.

- Definition through computational machines.
- Definition through constructional procedures (i.e. grammars).
- Definition through computable functions (i.e. recursion theory).

Church-Turing thesis (1934-1937)

At the level of Turing machines all rational models of computation are equivalent!

The fundamental questions

- Do natural hard problems exist?
- What are the consequences of answers YES or NO to this question?

- An investigation of our deduction process and our brain functionality is among the most original motivations for theory of computation, before any modern computer used to be available!
- This provides a strong link between "theory of computation" and "mathematical logic" as the two most basic and fundamental ways of providing finite presentation!

Theory of computation: complexity measures

Main Question: How much should we pay for a computation?

Typical cost functions

- Time
- Used memory
- even more complex cost functions!

The variable: is the length of the input!

Main objective: Study the behaviour of the cost function!

Acceptable (i.e. effective) cost functions

- Early days (1960's): Polynomially bounded functions.
- Nowadays: low-degree polynomially bounded.
- Trend (big data): $O(n \log n)$ or less!

The complexity class P (1960's: Cobham, Edmonds and Rabin)

Consists of all decision problems that can be solved by polynomial-time bounded deterministic decider algorithms.

It is good if we can effectively (i.e. fairly easily) solve a problem!

Theory of computation: P vs. NP

Fact

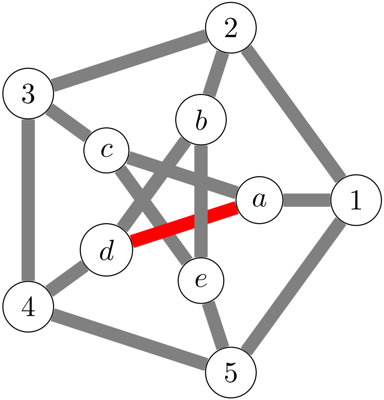
There exists a large number of fundamental decision problems (say more than 2000) for which any claim for a solution can be verified efficiently (i.e. in polynomial time), however, no efficient (i.e. polynomial time) solver is known for any one of these problems!

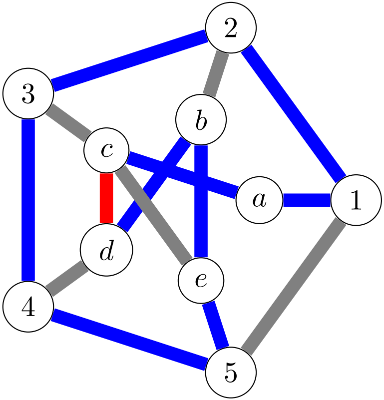
The complexity class NP (1970's: Cook, Levin, Karp)

Consists of all decision problems that can be solved by polynomial-time bounded nondeterministic acceptor algorithms.

NP-Complete: The class of hardest problems in NP.

Example for an NP-complete problem: Hamiltonian Cycle





An important question!

The fundamental questions

Do natural hard problems exist?

- One has to formulate easy and hard!
- We are not Zeus: hence it is generally believed that the answer is YES!
- It is astonishing that existence of hard problems is quite important in modern technological applications!

A fundamental problem: Do hard problems exist?

i.e., $P \stackrel{?}{=} NP$.

Why this is important!

- Natural hard problems are ideal primitives to be used when one needs unsolvable problems in a given context (e.g. cryptography!)
- If there is no natural hard problem then it means that we are ready to solve all given problems efficiently! Hence, one should be careful about the term “natural"!
- Think of an algorithm as a local discrete dynamical system on a discrete domain. Then connectivity of the domain is naturally related to the performance of the algorithm! Hence, existence of natural computationally hard problems somehow seems to be related to the non-existence of highly connected discrete structures of given fixed volume!

Theory of computation: 1 M$ Millennium Problems (Clay institute 2000)

Seven problems each gives you $1 million at least!

P vs. NP

Determine the answer to the question $P \stackrel{?}{=} NP$.

Riemann Hypothesis

The prime number theorem determines the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part 1/2.

There are 5 more problems and 4 more unsolved ones!

On randomness, proofs and computation

It seems that for the time being,

randomness

is our detour to handle our weakness in algorithm design!

Randomness and usefulness of hard problems

A couple of fundamental problems:

- Can one produce (i.e. simulate) almost ideal random bits?
- Is randomness useful in computation? Can one use randomness to get easier solutions?
- Can one reproduce a large number of random bits using a small number of ideal ones?
- What does Riemann Hypothesis say about the set of natural numbers?

On the concept of a "Proof"

How to make sure that a claim is true?

- A sound proof is a valid argument for the correctness of a claim.
- To make sure that a claim is true it is quite sufficient to have/see/verify a proof of it!
- However, to make sure that a claim is true it is also sufficient to make sure that there exists a proof of it!!!!

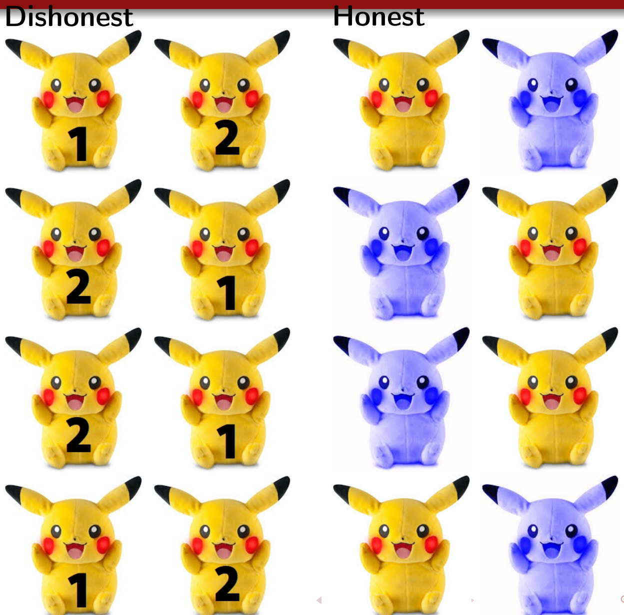
Example: the blind and the twins

Yellow Blue

Blue Yellow

Blue Yellow

Yellow Blue ✗ ✗



A challenging question!

- What if the blind is not supposed to hide his random bits!
- What if he is not also allowed to use randomness?

Example: BPP and secure communication

BPP is the randomized counterpart of P.

The class BPP

A language L is in BPP if there exists a randomized algorithm A such that

- $x \in L$ implies that $Pr(A(x) = accept) > 3/4$.
- $x \notin L$ implies that $Pr(A(x) = reject) > 3/4$.

A fundamental problem: Do hard problems exist?
i.e., $NP - BPP \stackrel{?}{=} \emptyset$.

Example: BPP and secure communication

In secure communication:

- It is assumed that everyone knows about the details of algorithms ENC and DEC except the security parameter κ (i.e. the key).
- The adversary problem: $\{p \mid \exists\ \kappa\ \ ENC(\kappa, p) = c\} \in NP$. (oversimplified!)

$NP - BPP = \emptyset \ \Rightarrow$ there is no secure communication system!

Study of a possible converse is the subject of modern provable cryptography!

Probabilistic algorithms

Question 1:

Does randomness fundamentally help in computation? i.e. are there problems with probabilistic polynomial-time algorithmic solutions but no deterministic one?

Question 2:

Does NP require strictly more than polynomial time?
i.e. natural hard problems do exist!

At least one of the answers is NO!!

Probabilistic algorithms

More on the story of random bits in computability!

On the existence of hard problems

Assuming factorization of integers has no efficient algorithm implies $P \neq NP$.

[Blum, Micali, Yao, Nisan, Impagliazzo, Wigderson]

Existence of hard problems (say $P \neq NP$ or something similar!) implies the existence of Pseudo-random generators.

Probabilistic proof system

A proof is an argument for a claim.
Main question: Is it valid?

$\exists$ a probabilistic verifier $V(claim, arg)$ for the claim, such that

- If the claim is true then $V(claim, arg^*) = true$ for some argument arg^*.
- If the claim is false then $V(claim, arg) = false$ for every argument arg with probability more than 0.99.

Probabilistic checkable proofs (PCP's)

$\exists$ a probabilistic verifier $V(claim, arg)$ for the claim, such that

the verifier only reads at most 10 bits of the argument at random.

[Arora-Safra, Arora-Lund-Motwani-Sudan-Szegedy-Hastad]

Every proof can be efficiently transformed into a PCP!

Applications

Assuming the existence of natural hard problems:

Non-approximability

Some NP-complete problems (e.g. MaxClique, MaxSat, ...) are non-approximable!

Grading answer sheets

There exists a randomized procedure using which one can grade the answer sheets of an exam in which one only reads at most 10 random characters from each sheet and the maximum probability of giving a wrong grade is less than 0.0001!

Authentication

There exists a randomize procedure using which you can prove to your bank on the Internet that you are YOURSELF without revealing your electronic signature at all!

On randomness,

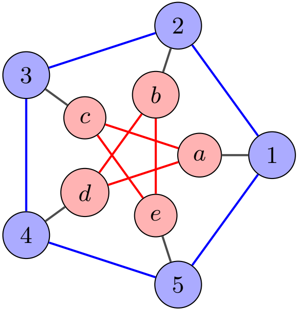
highly connected discrete structures

and Riemann Hypothesis

- Think of an algorithm as a local discrete-time dynamical system on a discrete domain (i.e. configuration space).
- The design problem is to construct such a system with a guarantee for fast access from starting configurations to accepting configurations using a limited number of operations!
- This definitely somehow is related to high connectivity of the domain in these specific directions!
- In a simplified randomized setting this turns into a global guarantee (e.g. think of a fast mixing Markov chain as a probabilistic algorithm!).
- Hence, the whole thing for maximum efficiency is somehow related to maximum connectivity of the configuration space (either for some specific directions or globally)!

Let us talk about maximally connected discrete structures of a fixed global measure.

Graphs and Matrices



	1	2	3	4	5	a	b	c	d	e
1	0	1	0	0	1	1	0	0	0	0
2	1	0	1	0	0	0	1	0	0	0
3	0	1	0	1	0	0	0	1	0	0
4	0	0	1	0	1	0	0	0	1	0
5	1	0	0	1	0	0	0	0	0	1
a	1	0	0	0	0	0	0	1	1	0
b	0	1	0	0	0	0	0	0	1	1
c	0	0	1	0	0	1	0	0	0	1
d	0	0	0	1	0	1	1	0	0	0
e	0	0	0	0	1	0	1	1	0	0

Random bits

A fundamental problem

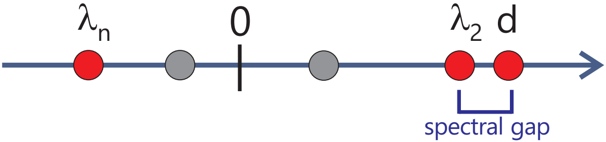
Design of pseudo-random generators, and extractors are among the most fundamental problems in ToC, Engineering and Science.

Highly connected Random regular graphs

A fundamental problem

How may one construct a highly connected graph on n vertices given only kn edges (for some constant k).

Random regular graphs: main questions



- What can be said about the spectral gap?
- What can be said about other connectivity related parameters as chromatic number, expansion, Hamiltonicity .....
- Analysis of the extremal cases are usually quite challenging problems.

Primes and Zeta Functions: definitions

- Consider a geometric space made of primes and their amalgams.
- e.g. $\mathbb{Q}$, number fields, function fields, Riemannian manifolds, graphs, ...
- The connectivity of the space is naturally related to number of primes and how they are mixed together.
- Connectivity is a fundamental concept that can be studied and measured in many different ways.
- A zeta (in general L) function is a mathematical concept that is supposed to present and reflect all these aspects in a reasonable way!

Primes and Zeta Functions

Rational numbers

The Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} (1 - p^{-s})^{-1}$ is related to Hecke operators but possibility for relation to a natural diffusion is not fully understood yet.

Graphs

The Ihara zeta function $\zeta(u) = \prod_{[P]\ prime} (1 - u^{\ell([P])})^{-1}$ is related to the adjacency operator and this relation is fully understood.

Apply $u := q^{-s}$ to compare!

Ramanujan Graphs

Riemann Hypothesis (RH) for $\mathbb{Q}$

If $\zeta(s) = 0$ and $0 < Re(s) < 1$ then $Re(s) = 1/2$.

Riemann Hypothesis (RH) for graphs (Ihara zeta func.)

If $\zeta(q^{-s})^{-1} = 0$ and $0 < Re(s) < 1$ then $Re(s) = 1/2$.

This is equivalent to the following:

Ramanujan graphs

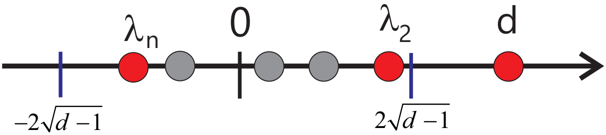
A $(q+1)$-regular graph with adjacency matrix A satisfies RH iff it is Ramanujan, i.e. if

$$\mu \overset{\text{def}}{=} \max\{|\lambda| \mid \lambda \in Spec(A) \ \& \ |\lambda| \neq q+1\}$$

then $\mu \leq 2\sqrt{q}$.

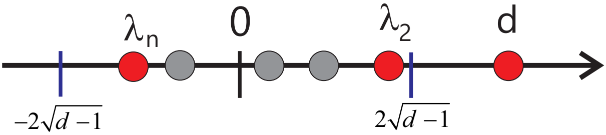
Spectrum of a Ramanujan Graph

Note: Regularity is $d = q + 1$.



- Nontrivial eigenvalues are small
- The graph is sparse but highly connected
- It is a good sparse approximation of a complete graph
- Alon-Boppana 1986: We can not beat the bound $2\sqrt{d-1}$ asymptotically
- The bound $2\sqrt{d-1}$ is the spectral radius of the infinite d-regular tree (i.e. the universal cover!)

Expanders



Expanders are sledgehammers of ToC! They are used in:

- Derandomization
- Complexity theory
- Error correcting codes
- Compressed sensing
- Communication networks
- Approximate counting
- Measure theory
- Number theory
- ...

Ramanujan graphs of arbitrary degree

A. W. Marcus, D. A. Spielman, N. Srivastava, 2013+
Published in *Annals of Mathematics* (2015)

There exist (arbitrarily large enough) bipartite regular Ramanujan graphs of arbitrary degree.

The proof is based on the fundamental technique of interlacing families of polynomials which is also used by the same authors to prove Kadison-Singer Problem.

Propaganda!

Euclid of Alexandria (about 300 BC):

There is no Royal Road to geometry.

Discrete structure zoo

Mathematics of discrete structures is a Royal Road to the heart of modern mathematics that is free to be used by any curious scholar!

The road passes through Computer Science land of Oz!

Some hot topics

- Big data and fast ($O(n \log n)$) algorithms.
- Machine learning and foundations of AI.
- Provable cryptography.
- Quantum computation.
- Theoretical biology.
- Network theory and modeling.
- Image processing and computer graphics.
- Compressed sensing
- Foundations of mathematics and programming.

What next?

Some speculations

- Technologically we are moving into a new era of networks! Hence, new paradigms of computability related to parallel and distributed computing are going to be important subjects in the field.
- Considering AI and its importance in our future life and technology, it is time to try to develop a sound and new theory of intelligent computation and intelligent algorithms!
- Brain has always been among the most sophisticated computers of all time! It seems that understanding how our brain works may lead to new paradigms for computability.
- Our daily need for huge data processing asks for ultra-fast efficient algorithms that are capable of processing huge datasets. This naturally asks for new techniques to design almost linear ultra-efficient algorithms as well as new techniques for their analysis.

A quotation!

In the prelude of "Récoltes et Semailles", Alexandre Grothendieck makes the following points on the search for relevant geometric models for physics and on Riemann's lecture on the foundations of geometry.

It must be already fifteen or twenty years ago that, leafing through the modest volume constituting the complete works of Riemann, I was struck by a remark of his "in passing".

... it could well be that the ultimate structure of space is discrete, while the continuous representations that we make of it constitute perhaps a simplification (perhaps excessive, in the long run ...) of a more complex reality; That for the human mind, "the continuous" was easier to grasp than the "discontinuous", and that it serves us, therefore, as an "approximation" to apprehend the discontinuous.

Thank you!

Comments are Welcome

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