The Metamathematics of Domain Theory and Beyond

Amir Daneshgar
Sharif University of Technology

http://www.sharif.ir/~daneshgar

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daneshgar@sharif.ir

[D. Hilbert, 1900]

Mathematical science is in my opinion an indivisible whole, an organism whose vitality is conditioned upon the connection of its parts.

Outline

1. Motivations
   - Theory of Computation
   - Nonlinear System Theory
   - Categorical System Theory

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   - Theory of Computation
   - Nonlinear System Theory
   - Categorical System Theory
2. Topology of Order Convergence
3. Some Categorical Results

Computation and fixed points (classical approach)

- An algorithm is a discrete-time discrete-space dynamical system on a state space $\Omega \subseteq \Gamma^*$ such that
  - The state space Ω of configurations is of infinite size.
  - There is a subset $I \subseteq \Omega$ with a one-to-one correspondence to the input space Σ^*.
  - The (one-clock) transition rule F has a finite description.
  - The (one-clock) transition rule F has local property.
  - Stable points (outputs) of this dynamics are the fixed points of F.

Main problem: How can we design such dynamics with a prescribed set of fixed points?

Computation and fixed points (functional approach)

- An algorithm is a combination of atomic functions and operations as
  - constant functions
  - assignments
  - (IF ... THEN .... ELSE ....) functions
  - LOOPING functions (e.g. WHILE .... DO ....)

  where these functions can be combined using combination of functions or recursion.

- Fact (Church-Turing): Such functions and operations are sufficient for computability (i.e. the design problem).
- Main problem: How can we describe such dynamics?

Denotational semantics

Informal definition

Denotational semantics is concerned with giving mathematical models of programming languages. Meanings for program phrases defined abstractly as elements of some suitable mathematical structure. (can be compared to operational and axiomatic semantics.)

Note: There are cases that we need recursive definitions!

The WHILE loop

The formal semantic equation

$$[While\ b\ Do\ c] = [If\ b\ Then\ (c; While\ b\ Do\ c)\ Else\ SKIP].$$

Hence, one should look for a function f satisfying the following recursion:

The fixed point equation

$$F(f) = Cond([b], ([c]; f), id).$$

Note: There are more complex recursive equations! Think of $[D \longrightarrow D] = [D]$!

Some other fixed-point-related topics

[Scott (1969) - America, Rutten (1987) - Wagner (1994) - Flag, Kopperman (1995)]

Generalizations of Scott's inverse limit theorem in domain theory and denotational semantics.

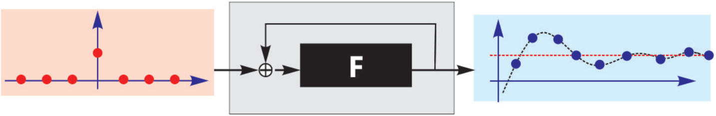
[Abramsky, Gay, Nagarajan (1995) - Yeasin (2012)]

Fixed points of linear functors and concurrent programming.

[Freudenthal (1937) - ongoing!]

Stabel homotopy theory.

Fixed points (summary)



Fixed points of endofunctors

We need to somehow make sure that some sort of limits of endofunctors of a monoidal category as

$$\lim_{n\to\infty} F^n(x)$$

exist!

Just one more motivating example

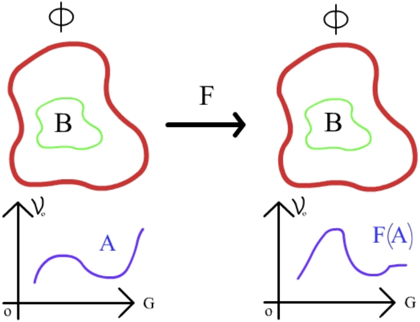
The mean operator

You are asked to compute the mean of three natural numbers a, b and c. That is you must compute

$$\frac{a+b+c}{3}.$$

What if you are asked to do this only using addition and comparison? (This means that you are not allowed to use division!)

I/O SYSTEM THEORY: BASIC IDEA



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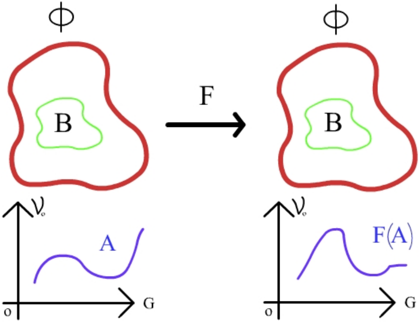
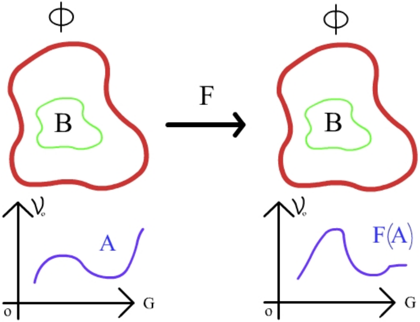
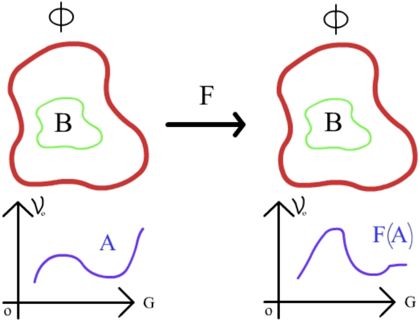


- Φ is a function space with some nice algebraic and topologic properties (usually completeness conditions).



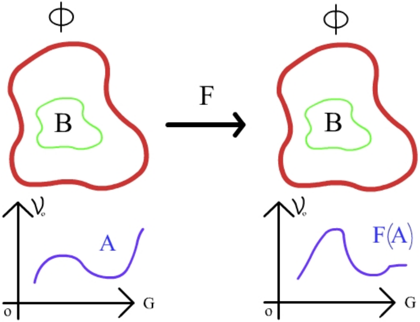
- Specially Φ is reconstructable, i.e. can be reconstructed properly by a (relatively small) subspace B of its generic objects.

I/O SYSTEM THEORY: BASIC IDEA



- Usually Φ is chosen to be a product space $G \times \mathcal{V}_0$.
  e.g. SPEACH: $G = R$ is the time and $\mathcal{V}_0 = R$ is the space of levels.
  e.g. IMAGE: $G = Z^2$ is the two dimensional space $\mathcal{V}_0 = Z$ is the space of gray-scales.

I/O SYSTEM THEORY: BASIC IDEA



- F is a natural map (i.e compatible with the structure), with nice representation properties.

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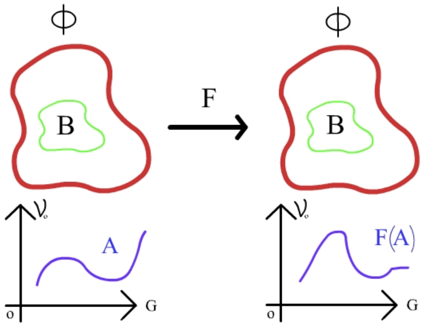

- The whole setup should be in coherence with natural phenomena and be able to simulate input-output behaviour of such systems (this is called I/O system theory).

A classical example: LSI systems

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- F is linear and shift invariant, i.e

$$F(A + \alpha B) = F(A) + \alpha F(B),$$

$$F(T_g(A)) = T_g(F(A)),$$

where T_g is the translation-by-g operator, i.e.

$$T_g(A)(t) = A(t - g).$$

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$$T_g(A)(t) = A(t - g).$$

- We have the following reconstruction:

$$F(A)(t) = Conv(A, \delta_F) = \int_{-\infty}^{+\infty} \delta_F(\tau) A(t - \tau) d\tau.$$

LSI systems (example)

Consider the discrete system

$$T(f)(n) \stackrel{\text{def}}{=} \frac{1}{3}(f(n) + f(n-1) + f(n-2))$$

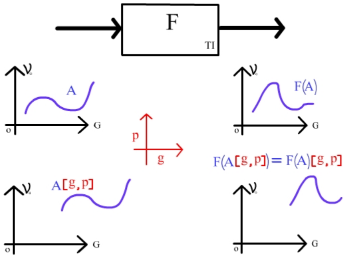
as a smoother (low-pass filter) on discrete signals $f : \mathbf{Z} \longrightarrow \mathbf{R}$. The first important observation is that T is an LSI system and its impulse response h is the following:

$$h(n) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{3} & n = 0, 1, 2 \\ 0 & n \neq 0, 1, 2. \end{cases}$$

Therefore, T can be expressed as the convolution

$$T(f)(n) = \sum_{m=0}^{\infty} f(n-m)h(m) = \frac{1}{3}(f(n)+f(n-1)+f(n-2)).$$

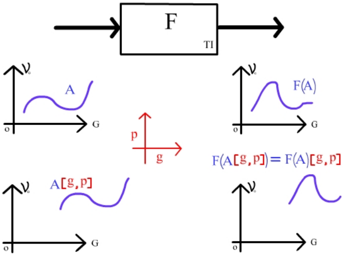
Translation invariance



- Consider

$$A = \{(t, A(t)) \ \mid \ t \in G\}.$$

Translation invariance



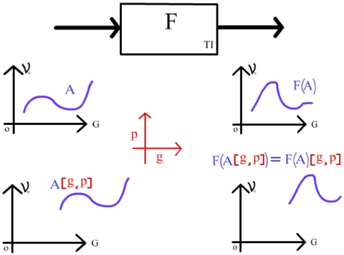
- Then we define the translation operator as follows,

$$A[g,p] = \{(t+g, A(t)*p) \ | \ t \in G\},$$

which means,

$$A[g,p](t) = A(t-g)*p.$$

Translation invariance



- An operator F is translation invariant if

$$F(A[g,p]) = F(A)[g,p].$$

Note: For an LSI operator, translation invariance on the range is equivalent to DC-gain 1.

Operators v.s. Comparison

- The set $A = \{(t, A(t)) \mid t \in G\} \subseteq G \times \mathcal{V}_0$ can be considered as a function or a fuzzy set.

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Operators v.s. Comparison

- The set $A = \{(t, A(t)) \mid t \in G\} \subseteq G \times \mathcal{V}_0$ can be considered as a function or a fuzzy set.
- The difference between the two points of view is in the way we look at the range $\mathcal{V}_0$.
- The functional approach is when we consider algebraic properties and the fuzzy set approach is when we consider the comparative structure (order structure) of $\mathcal{V}_0$.
  e.g. The neutral element for summation is 0, while the neutral element for the supremum is $-\infty$.

Minkowski addition and subtraction

- Let $(G, +, -, 0)$ be a group and $\mathcal{V}_0 = (\Omega, \leq, *, \div, 0) \cup \{-\infty, +\infty\}$ be a lattice ordered group with the universal bounds $-\infty$ and $+\infty$ such that,

$$\div(-\infty) = +\infty, \qquad \div(+\infty) = -\infty,$$

$$(-\infty) * (+\infty) = (+\infty) * (-\infty) = (+\infty) * (+\infty) = (+\infty)$$

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$$\forall\ p \in \Omega \quad (+\infty) * p = p * (+\infty) = +\infty,$$

$$\forall\ p \in \Omega \quad (-\infty) * p = p * (-\infty) = -\infty.$$

Minkowski addition and subtraction

- Minkowski addition and subtraction for L-fuzzy sets are defined as follows

$$A \oplus B = \sup_{g} A[g, B(g)] \quad , \quad A \ominus B = \inf_{g} A[g, \div B(g)].$$

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- e.g. Let
  $G =< \ a \mid \ 3a = 0 \ >$,
  $A = \{(0, x_1), (a, x_2), (2a, x_3)\}$,
  $B_1 = \{(0, 0), (a, 0), (2a, -\infty)\}$.
  Then,

$$A \oplus B_1 = \{(0, \sup(x_1, x_3)), (a, \sup(x_2, x_1)), (2a, \sup(x_3, x_2))\}$$

Minkowski erosion as convolution

- Minkowski erosion is defined as follows

$$Er(A, B) = A \ominus B^s.$$

where,

$$B^s = \{(-x, B(x)) \mid x \in G\}.$$

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- Minkowski erosion is non-linear and behaves as a convolution operator.

The kernel

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  $G =< \ a \mid \ 3a = 0 \ >$,
  $A = \{(0, x_1), (a, x_2), (2a, x_3)\}$,
  $F(A) = \{(0, m), (a, m), (2a, m)\}$, where
  $m = Median(x_1, x_2, x_3)$.
  Then,

$$B(F) = \{\{(0,0), (a,0), (2a, -\infty)\}, \{(0, -\infty), (a, 0), (2a, 0)\},$$

$$\{(0,0), (a, -\infty), (2a, 0)\}\}$$

The reconstruction theorem

- A TI operator F is isotone if

$$\forall\ A, B \quad A \leq B \ \Rightarrow\ F(A) \leq F(B).$$

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- **Strong Reconstruction Theorem** (D. 1995)
  Let F be an isotone TI operator. Then

$$F(A) = \sup_{D \in K(F)} Er(A, D);$$

  and if the base of F exists then

$$F(A) = \sup_{B \in B(F)} Er(A, B).$$

Example 1: The mean filter

Again consider the discrete system

$$T(f)(n) \stackrel{\text{def}}{=} \frac{1}{3}(f(n) + f(n-1) + f(n-2)).$$

One may note that T is an isotone TI system with the basis consisting of all maps $h_{r,s}$, $(r, s \in \mathbf{R})$ defined as follows:

$$h_{r,s}(n) \stackrel{\text{def}}{=} \begin{cases} -r-s & n=-2 \\ s & n=-1 \\ r & n=0 \\ -\infty & otherwise. \end{cases}$$

Hence,

$$\begin{aligned} T(f)(n) &= \sup_{r,s\in\mathbf{R}} Er(f, h_{r,s})(n) \\ &= \sup_{r,s\in\mathbf{R}} \inf_{m\in\mathbf{Z}} (f(n+m) - h_{r,s}(m)) \end{aligned}$$

Example 2: The median filter

- Let $G =< \ a \mid \ 3a = 0 \ >$,
  $A = \{(0, x_1), (a, x_2), (2a, x_3)\}$,
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- Strong reconstruction theorem implies that
  $Median(x_1, x_2, x_3) = F(A)(0)$
  $= \sup(\inf(x_1, x_2), \inf(x_2, x_3), \inf(x_3, x_1))$.

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  $= \sup(\inf(x_1, x_2), \inf(x_2, x_3), \inf(x_3, x_1))$.
- This is a fuzzification of the Boolean expression
  $x_1x_2 + x_2x_3 + x_3x_1$.

Main objective

Study the space of endofunctors of nice monoidal categories.

The above scenario contains:

- (Analysis) is the study of fixed (i.e. stable) points of endofunctors.
- (Design) is the study of nicely presentable endofunctors which is essentially the study of nice accessible subcategories of the corresponding functor category.

Categorical system theory

Main objective

Studies the space of endofunctors of nice monoidal categories.

In what follows we talk about:

- Both problems strongly rely on the existence of nice categorification of basic topological concepts as the limit.
- The design problem is tightly linked to the theory of integration.
- Convolution is essentially an enriched HOM functor.

Categorical system theory

Main objective

Studies the space of endofunctors of nice monoidal categories.

When the monoidal category is a G-graded category for a group G:

- The study essentially reduces to the representation theory of G.
- This study is closely related to cohomology of groups, elliptic homotopy theory, noncommutative Fourier transforms, and geometric Langlands program.
- In this regard:
  The basic atomic concept is a categorification of the convolution operation.

Basic questions so far

- How can we guarantee and in the best case compute the (canonical) fixed points of a given endofunctor of a nice monoidal category?
- How can we define a generalized convolution on a product (or more generally on a sheaf) category?

The central question!

It is interesting to note that both questions are related to "How can we define a generalized topological convergence structure on a given category?"

Basic concepts (reminder)

- A domain is a continuous dcpo (eq. a dcpo with a basis).
- A subset D of a preorder L is directed if every pair of elements in D has an upper bound in D.
- A net on a set X is a function $N : J \longrightarrow X : j \longmapsto x_j$ such that J is a directed set.
- A function $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ on a metric space X is said to be lower semicontinuous if for any sequence x_j converging to $x \in X$ we have

$$f(x) \leq \sup_j \inf_{k \geq j} f(x_k) \stackrel{\text{def}}{=} \liminf f(x_j).$$

What if X is a complete lattice or a dcpo?

- Consider the collection of nets $\{((x_j), x) \mid x \leq \liminf x_j\}$.
- Basic question: When is this collection topological?

Erne, Hoffmann, Lawson, Markowsky, ...

- On a complete lattice the collection is topological if and only if the lattice is continuous.
- On a dcpo the collection is topological if the dcpo is continuous (i.e. it is a domain), and the induced topology is the Scott topology.

Note: For the case of dcpo's the definition must be in terms of eventual lower bound.

Order convergence

Definition

Let $\{x_\alpha\}_{\alpha \in \Delta}$ be a net on the poset $(X, \leq)$. we define

$$\mathsf{L}_\alpha \stackrel{\text{def}}{=} \{l \in X \ : \ l \leq \{x_\beta \ : \ \beta \geq \alpha\}\},$$

$$\mathsf{U}_\alpha \stackrel{\text{def}}{=} \{u \in X \ : \ u \geq \{x_\beta \ : \ \beta \geq \alpha\}\}.$$

We say that $\{x_\alpha\}_{\alpha \in \Delta}$ order converges to x if and only if $\sup(\bigcup_\alpha \mathsf{L}_\alpha)$ and $\inf(\bigcup_\alpha \mathsf{U}_\alpha)$ both exist and are equal to x. In this case we write $x_\alpha \stackrel{o}{\to} x$.

Some facts

Consistency

Let $\{x_\alpha\}_{\alpha\in\Delta}$ be a net on $(X,\leq)$. If expressions $\inf_\alpha(\sup_{\beta\geq\alpha}\{x_\beta\})$ and $\sup_\alpha(\inf_{\beta\geq\alpha}\{x_\beta\})$ are both eventually meaningful then,

$x_\alpha \xrightarrow{o} x$ if and only if

$$\inf_\alpha(\sup_{\beta\geq\alpha}\{x_\beta\}) = \sup_\alpha(\inf_{\beta\geq\alpha}\{x_\beta\}) = x.$$

Some facts

Categorification

- Why the convergence structure should be topological?
- At the semantic level we need a convergence structure on a category of semantics (types).
- Can the concept of order convergence be categorified properly?

K. R. Wagner (1994)

The concept of order convergence for sequences can be categorified good enough to let us prove a generalized Scott's inverse limit theorem in a quantale enriched setting!

Note: How farther can we go?

Categorical prerequisites I

- Let $\mathcal{V} = (\mathcal{V}, ., \div, I, a, l, r, c)$ be a symmetric closed monoidal category enriched over itself, with identity I, for which $V.- \dashv - \div V$ for any $A \in V$ and the base category $\mathcal{V}_0$ is complete and cocomplete.

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- We also assume that V $= [I, -]_{\mathcal{V}_0} : \mathcal{V}_0 \longrightarrow$ Set is the base functor and we adapt the multiplicative notation in $\mathcal{V}$.
- On the other hand, let $\mathcal{D}$ be a class of diagram–schemes and assume that $\mathcal{V}_f \subseteq \mathcal{V}_0$ is a small full subcategory such that $\mathcal{V}_0$ is a free $\mathcal{D}$–cocompletion of $\mathcal{V}_f$ in the sense that,
  - The totality of all $\mathcal{D}$–colimits constitute a density presentation for the inclusion i : $\mathcal{V}_f \hookrightarrow \mathcal{V}_0$.
  - For any $A \in \mathcal{V}_f$, the hom–functor $[A, -]$ preserves all $\mathcal{D}$–colimits.

Categorical prerequisites II

- we consider a (small) commutative group $(G, +, -)$, with identity 0 and we focus on the product space $\Phi = \mathcal{V}_0^G$ equipped with the pointwise categorical structure of $\mathcal{V}_0$.

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- It is easy to see that, considering Φ as a category of functors between the discrete category G and $\mathcal{V}_0$, Φ inherits the completeness properties of $\mathcal{V}_0$ and that it has a dense $\mathcal{D}$–presentable full subcategory Φ_f.

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- Hereafter, $\Delta_W \in \Phi$ is the constant functor with value W.

Two functors

- Consider the following maps for a fixed $D \in \Phi$,

$$\mathsf{T}_D : \mathrm{obj}(\Phi) \longrightarrow \mathrm{obj}(\Phi),$$

$$\mathsf{T}_D(A)_{(z)} = \prod_{|J|<\infty} \Big(\bigoplus_{j\in J} W_j\Big)^{[A_{(x)}, \oplus_j (\Delta_{W_j} \div D_{(-x+z)})]_\Phi},$$

$$\mathsf{H}_D : \mathrm{obj}(\Phi) \longrightarrow \mathrm{obj}(\Phi),$$

$$\mathsf{H}_D(B)_{(z)} = \coprod_{|J|<\infty} {}^{[\oplus_j (\Delta_{V_j} \cdot D_{(-z+x)}), B_{(x)}]_\Phi} \Big(\bigoplus_{j\in J} V_j\Big),$$

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- **Theorem**(D.& Hashemi 2000)
  Both T_D and H_D can be naturally extended to define functors on Φ for any $D \in \Phi$.

An important special case

- If $\bigoplus$ is preserved by $(\div)$ (resp. $(.)$) then

$$\mathsf{T}_D(A)_{(z)} = \coprod_{x \in G} A_{(x)} . D_{(-x+z)} \stackrel{\text{def}}{=} \dot{\mathsf{T}}_D(A)_{(z)}$$

$$(resp.\ \ \mathsf{H}_D(B)_{(z)} = \prod_{x \in G} B_{(x)} \div D_{(-z+x)} \stackrel{\text{def}}{=} \dot{\mathsf{H}}_D(B)_{(z)}).$$

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- **Theorem**(D.& Hashemi 2000)
For any $D \in \Phi$ we have $\dot{\mathsf{T}}_D \dashv \dot{\mathsf{H}}_D$ (i.e. $\dot{\mathsf{T}}_D$ is the left adjoint of $\dot{\mathsf{H}}_D$).

An important special case

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- **Theorem**(D.& Hashemi 2000)
  There exist natural isomorphisms $\tilde{a}, \tilde{l}, \tilde{r}$ and $\tilde{c}$ such that for any $D \in \mathrm{obj}(\Phi)$, $\dot{\Phi} = (\Phi, \dot{\mathsf{T}}_D, \dot{\mathsf{H}}_D, P^{0,I}, \tilde{a}, \tilde{l}, \tilde{r}, \tilde{c})$ is a symmetric closed monoidal category.

General reconstruction theorem

- A point $P^{g,V} \in \Phi$ with value $V \in \mathrm{obj}(\mathcal{V}_0)$ at coordinate $g \in G$ is defined (up to isomorphism) as

$$P^{g,V}(x) \stackrel{\mathrm{def}}{=} \begin{cases} V & x = g \\ -\infty & x \neq g. \end{cases}$$

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- **Theorem**(D.& Hashemi 2000)
  Let $\mathsf{F} : \dot{\Phi} \longrightarrow \dot{\Phi}$ be a $\dot{\Phi}$–functor such that F preserves the internal Hom of $\dot{\Phi}$, $\dot{\mathsf{H}}_D$, for any $D \in \Phi$; and $Ker(\mathsf{F})$ is a $\mathcal{D}$–type diagram–scheme. Then F has a representation as a $\mathcal{D}$–colimit of representables as

$$\mathsf{F}(A) \simeq \mathrm{Colim}_{(D,d)\in Ker(F)} \dot{\mathsf{H}}_D(A).$$

Example I

- Consider $G = (\mathbf{R}, +)$ and $\mathcal{V} = (\mathbf{R}, +, \leq)$ with two universal bounds $+\infty$ and $-\infty$. Then $\dot{\mathsf{T}}$ is the Minkowski addition and $\dot{\mathsf{H}}$ is the erosion operator.

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- Note that in this case the general reconstruction theorem yields the classical reconstruction theorem for TI operators.

The uniform case

- Now consider the following maps for a fixed $D \in \Phi$,

$$\tilde{\mathsf{T}}_D : \Phi \longrightarrow \mathcal{V},$$

$$\tilde{\mathsf{T}}_D(A) = \prod_{|J|<\infty} \Big(\bigoplus_{j\in J} W_j\Big)^{[A_{(x)}, \oplus_j (\Delta_{W_j} \div D_{(-x+z_j)})]_\Phi},$$

$$\tilde{\mathsf{H}}_D : \Phi \longrightarrow \mathcal{V},$$

$$\tilde{\mathsf{H}}_D(B) = \coprod_{|J|<\infty} {}^{[\oplus_j (\Delta_{V_j} . D_{(-z_j+x)}), B_{(x)}]_\Phi} \Big(\bigoplus_{j\in J} V_j\Big),$$

where $\bigoplus$ is a suitable associative bifunctor.

A simple case

- As a simple case we have,

$$\ddot{\mathsf{T}}_D(A) = \prod_j W_j^{[A_{(x)}, {}^{\Delta}W_j \div D_{(-x+z_j)}]_\Phi} = \prod_z \dot{\mathsf{T}}_D(A)_{(z)},$$

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- **Theorem**(D.& Hashemi 2000)
  If the tensor of $\mathcal{V}$ is preserved by coproducts, then there exist a natural composition law and an identity map such that $[D, B] \overset{\text{def}}{=} \ddot{\mathsf{H}}_D(B)$, as the internal Hom, turns Φ into a $\mathcal{V}$–category.

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- Can you prove a similar theorem for the general case?

Example II

- Let $G = (\mathbf{R}, +)$ and $\mathcal{V} = (\mathbf{R}^+, ., \geq)$. Then

$$\ddot{\mathsf{H}}_D(B) = \inf_z \left(\sup_x \left(B_{(x)} \div D_{(-z+x)}\right)\right)$$

in the ordinary order of real numbers.

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- Let $\oplus = +$ be the ordinary addition of real numbers. Then,

$$\tilde{\mathsf{H}}_{D}(B) = \inf_{c_j, z_j} \{\Sigma_j\, c_j \mid\ \forall\, x\ \ B_{(x)} \leq \Sigma_j\, c_j D_{(-z_j+x)}\} = (B : D)$$

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- This is the Haar fraction!

Some after-thoughts

Having a nice convergence structure on a complete closed monoidal category may help to

- Prove new fixed point theorems in the context of denotational semantics.
- Define new and categorical-valued integrals.
- Define noncommutative Forier transforms at a categorical level.

To do this one may use her/his intuition from

- Domain theory.
- Non-linear system theory.
- Homotopy theory.
- Geometric Langlands program.

Epilogue

The ultimate aim is a flexible and general theory of enriched sheaves!

Thank you!

Comments and Criticisms are Welcomed

daneshgar@sharif.ir