On Homomorphisms to the Five-Cycle

Amir Daneshgar
Sharif University of Technology

http://www.sharif.ir/~daneshgar

May 2015 (Ordibehesht 1394)

daneshgar@sharif.ir

Outline

Chromatic number behaviour

Hardness

Graph coloring problem is among the hardest computational problems we tackle.

A consequence of the PCP theorem

Hastad *et.al.* (1992-1996): The graph colouring problem is non-approximable unless $\mathbf{NP} = \mathbf{RP}$!!!

Main question

How does the chromatic number $\chi(G)$ behave with respects of the structural properties of G?

A conjecture [B. Reed 1998]

$\chi(G) \leq \lceil \frac{1}{2}(\Delta + \omega + 1) \rceil$.

Chromatic number under sparsity I

Main question

How does χ behave under structural sparsity conditions?

Reed's conjecture for triangle-free graphs

$\chi(G) \leq \frac{\Delta}{2} + 2.$

Some facts

- A. Johansson (1996): For triangle-free graphs with large Δ we have $\chi(G) \leq O(\frac{\Delta}{\log \Delta})$.
- Conjecture is true for $\Delta \leq 4$.

Chromatic number under sparsity II

Main question

What if one imposes stronger sparsity conditions?

Imposing large girth conditions

- [A. V. Kostochka (1978)]: If G has large girth and $\Delta(G) \geq 5$ then $\chi(G) \leq \frac{\Delta}{2} + 2$.

The first open questions!

A: Does there exists a 5-chromatic triangle-free graph with $\Delta \leq 5$?

B: Does there exists a 6-chromatic triangle-free graph with $\Delta \leq 6$?

- B seems to be the easiest open case!

Chromatic number under sparsity III

Kostochka-Reed parameter

Let $\mathcal{G}(d, g)$ be the class of all d-regular graphs of girth larger than g. Define,

$$KR(d) \stackrel{\text{def}}{=} \inf_{g>3} \sup_{G\in\mathcal{G}(d,g)} \chi(G).$$

[M. Madani (2015)]

Let $\mathcal{G}'(d, g)$ be the class of all graphs of maximum degree d and of girth larger than g. Then,

$$KR(d) = \inf_{g>3} \sup_{G\in\mathcal{G}'(d,g)} \chi(G).$$

Chromatic number under sparsity III

Triviality

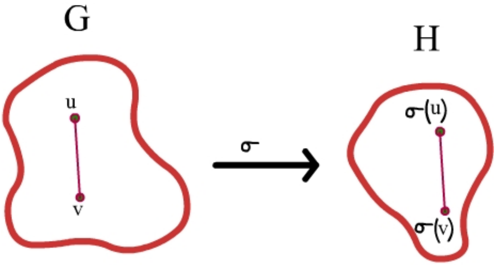
If $\Delta(G) \leq 4$ then Reed's conjecture is not interesting at all!

Idea

What if we impose more constraints on the coloring problem itself?

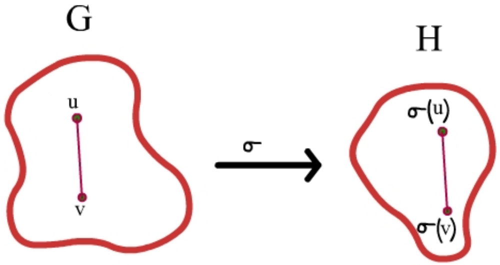
Let's talk about this!

A graph homomorphism



A graph homomorphism

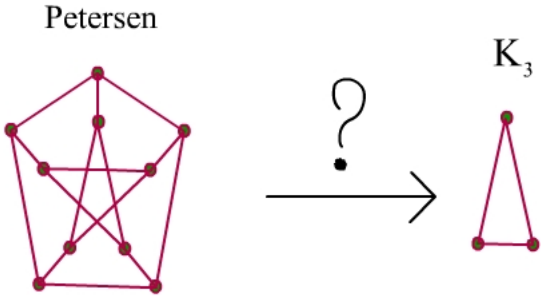
A graph homomorphism



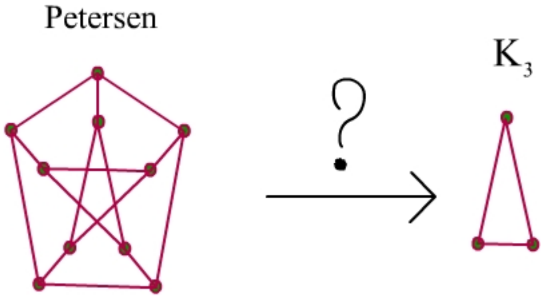
A graph homomorphism

- A graph homomorphism σ from a graph G to a graph H is a map $\sigma : V(G) \longrightarrow V(H)$ such that $uv \in E(G)$ implies $\sigma(u)\sigma(v) \in E(H)$.

A question

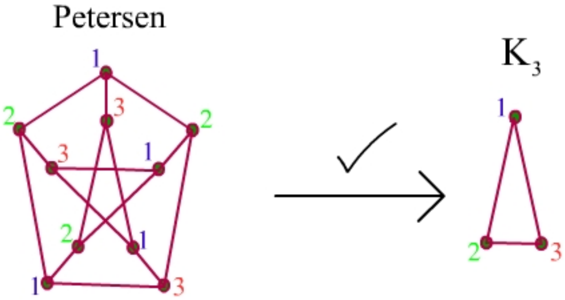


A question

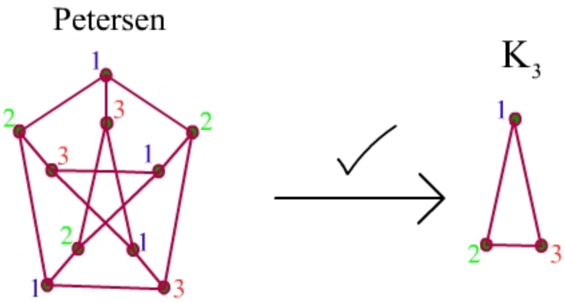


- Does there exist a homomorphism from the Petersen graph to the triangle K_3?

Graph colouring

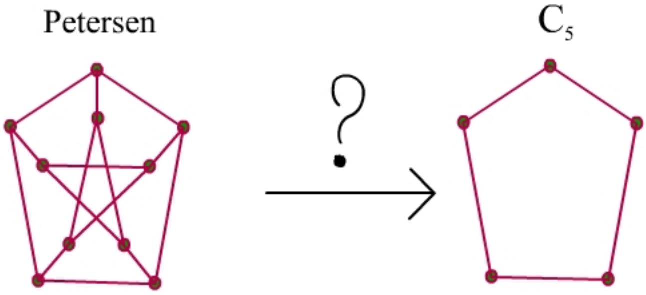


Graph colouring

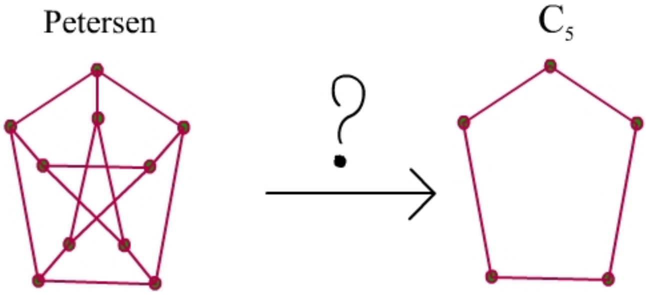


- Homomorphisms to K_n is equivalent to colouring the vertices of the graph by n colours such that the terminal ends of each edge have different colours.

Another question!



Another question!



- Does there exist a homomorphism from the Petersen graph to the 5-cycle C_5?

Circular chromatic number

- The circular complete graph, $K_{\frac{n}{r}}$, has the vertex set

$$\{0, 1, \cdots, n-1\}$$

and the edge set

$$\{ij \mid r \leq |i-j| \leq n-r\}.$$

- The circular chromatic number of a graph G is defined as

$$\chi_c(G) \stackrel{\text{def}}{=} \inf\{\frac{n}{r} \mid G \longrightarrow K_{\frac{n}{r}}\}.$$

- Example: $K_{\frac{5}{2}} = C_5$.

Girth-closed classes

Girth-closed and odd girth-closed classes

A class of simple graphs $\mathcal{G}$ is said to be girth-closed (resp. odd-girth-closed) if for any positive integer g there exists a graph $G \in \mathcal{G}$ such that the girth (resp. odd-girth) of G is greater than or equal to g.

Pentagonal and odd-pentagonal classes

A girth-closed (resp. odd-girth-closed) class of graphs $\mathcal{G}$ is said to be pentagonal (resp. odd-pentagonal) if there exists a positive integer g^* depending on $\mathcal{G}$ such that any graph $G \in \mathcal{G}$ whose girth (resp. odd-girth) is greater than g^* admits a homomorphism to the five cycle (i.e. is C_5-colorable).

The pentagon problem

J. Nešetřil (1999)

Is the class of simple 3-regular graphs pentagonal?

Nesetril parameter

Let $\mathcal{G}(d,g)$ be the class of all d-regular graphs of girth larger than g. Define,

$$Nes(d) \stackrel{\text{def}}{=} \inf_{g>3} \sup_{G\in\mathcal{G}(d,g)} \chi_c(G).$$

Pentagon problem

Is it true that $Nes(3) \leq 2.5$?

Chromatic number under sparsity III

Let's try to show that,

Objective

A lower bound for $KR(d)$ implies a lower bound for $Nes(d)$.

Some negative results

- [A. V. Kostochka, J. Nesetril, P. Smolikova (2001)]
  If C_5 is replaced by C_{11}, then a similar conjecture does not hold.
- [I. M. Wanless and N. C. Wormald (2001)]
  If C_5 is replaced by C_9, then a similar conjecture does not hold.
- [H. Hatami (2005)]
  If C_5 is replaced by C_7 then a similar conjecture does not hold.

Some positive results

- [Brooks' theorem] If C_5 is replaced by C_3, then a similar conjecture does hold.
- [A. Galluccioa *et.al.* (2001)] For every fixed simple graph H the class of H-minor free graphs is pentagonal.
- [O. V. Borodin *et.al.* (2004)] The class of planar graphs, projective planar graphs, graphs that can be embedded on the torus or Klein bottle are pentagonal.
- [O. V. Borodin *et.al.* (2008)] The class of simple graphs as G for which every subgraph of G has average degree less than $12/5$, is pentagonal (actually with $g^* = 3$).

The odd-girth constraint

Negative results

- [M. Gebleh (2007)]
  There exists an odd-girth-closed subclass of simple 3-regular graphs (i.e. spiderweb graphs) which is not odd-pentagonal. (Actually, the circular chromatic number of any spiderweb graph is equal to 3.)

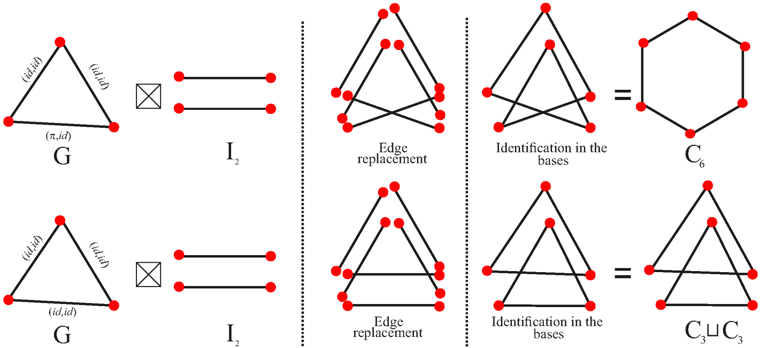
Positive results

- [D., M. Madani (2015)]
  Let $\mathcal{C}$ be the subclass of the class of generalized Petersen graphs for which one of the following conditions hold.

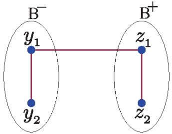
  (a) $\mathrm{Pet}(n,k)$, where k is even, n is odd and $n \overset{k-1}{\equiv} \pm 2$.
  (b) $\mathrm{Pet}(n,k)$, where both n and k are odd and $n \geq 5k$.

  Then $\mathcal{C}$ is odd-pentagonal.

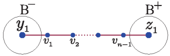
Random 2-lifts



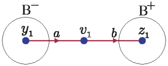
Examples for cylinders



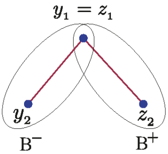
The ⊓-cylinder,



Path cylinder P_n



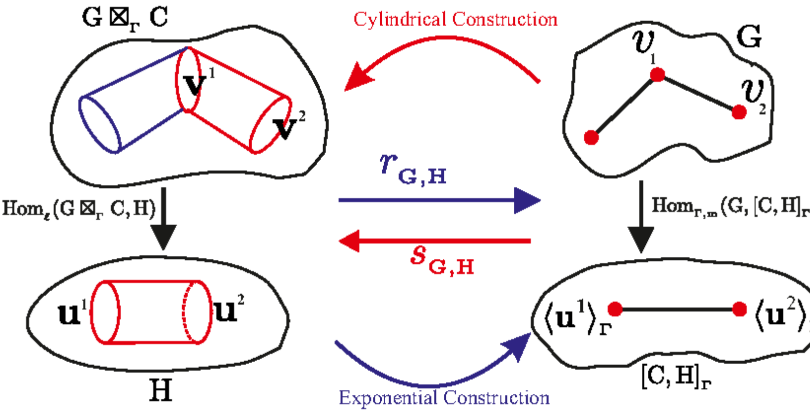
Path cylinder $\vec{P}_2$ of length two,



The looped line-graph cylinder.

The Petersen graph Show

Schematic duality diagram



Powers and subdivitions

Definitions

- The graph $G^{\frac{1}{t}}$ (i.e. the t-subdivision of G) is obtained by replacing each edge of G by a path of length t.
- The kth power functor on graphs is the right adjoint to the kth subdivision functor and for a graph G yields a graph G^k on the same vertex set where $u \sim v$ if there exists a walk of length k between u and v in G.
- For any graph G, define $G^{\frac{k}{t}} \stackrel{\text{def}}{=} (G^{\frac{1}{t}})^k$.

Note: If $G \longrightarrow H$ then for any integer $k > 1$ we have $G^k \longrightarrow H^k$.

Examples and applications

Examples

- $C_5^3 = K_5$.
- $Pet^3 = K_{10}$.
- For any integer $t \geq 1$ we have $(K_n)^{\frac{6t+1}{2t+1}} = K_{tn^2-tn+n}$.

Applications

- Petersen $\not\to C_5$.
- Coxeter $\not\to C_7$.

An implication of Pentagon problem (if the answer is YES)

For any integer g there exists a 3-regular graph of girth larger than g whose third power is 5-colorable.

Using tree-cylinders one can prove:

[D., M. Madani (2015)]

Let

- H be a graph with odd girth at least $2k+3$,
- G be a $d(d-1)^k$-regular graph with girth g and odd girth og, where $G \not\to H^{2k+1}$,

then there exist (many non-isomorphic) d-regular graphs G' such that

- $girth(G') \geq g$,
- $oddgirth(G') \geq og$,
- and $G' \not\to H$.

A connection to Reed's conjecture

A useful corollary

Let

- $(2k+1)r - nk > 0$,
- G be a $d(d-1)^k$-regular graph with girth g and odd girth og, where $\chi_c(G) > \frac{n}{(2k+1)r-nk}$,

then there exist (many non-isomorphic) d-regular graphs G' s.t.

- $girth(G') \geq g$,
- $oddgirth(G') \geq og$,
- and $\chi_c(G') > \frac{n}{r}$.

A consequence

Summary!

$$KR(d(d-1)^k) > \lceil \frac{n}{(2k+1)r - nk} \rceil \;\Rightarrow\; Nes(d) > \frac{n}{r}.$$

Let $d = 3$, $n = 5$, $r = 2$ and $k = 1$. Then, using this result,

A failure!

Existence of 6-regular 6-chromatic graphs of large girth would have disproved the pentagon problem!

But this is not the case!!!!

A second attempt

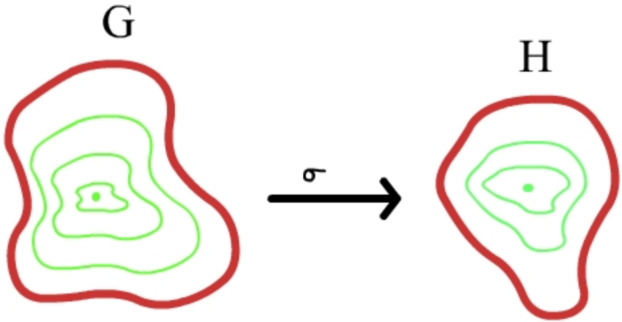
Let $d = 3$, $n = 7$, $r = 3$ and $k = 2$. Then, using this result,

A problem

$7 < KR(12)$ implies $\frac{7}{3} < Nes(3)$.

This is H. Hatami's result!

Basic idea



- If G is homomorphic to H then H must be much more connected than G.
- (Main question): How can we construct maximally connected sparse 3-regular graphs?!

Random 2-lifts

Let G be a $2d$-regular graph and randomly assign labels to the edges so that at each vertex we have exactly d flips.

Main question

What is the maximum connectivity of such π-lifts?

A construction and a problem

Let $n = 2^t + 2$, start from K_{n+1} and construct t maximally connected random π-lifts recursively.

Does the resulting 3-regular graph admits a homomorphism to C_5?

We are trying to verify!!!

Thank you!

Comments and Criticisms are Welcomed

daneshgar@sharif.ir