20.3 Exercises

20.3.1. If R and S are two relations show that $dom(R^{-1}) = ran(R)$ and $ran(R^{-1}) = dom(R)$. Also, show that if $R \subseteq S$ then $S^{-1} \subseteq R^{-1}$.

20.3.2. Define "one to one", "onto" relations.

20.3.3. Let R be an equivalence relation and define $|R|$ to be the number of equivalence classes of R. Then show that if for two equivalence relations R and S we have $R \subseteq S$, then $|S| \subseteq |R|$

20.3.4. Let R be an equivalence relation on a set A. Then a subset $B \subseteq A$ is said to be R-*saturated* if for all a and b in A, the statements $a \in B$ and $(a, b) \in R$ imply $b \in B$. Show the correctness of the following statements.

- A subset B is R-saturated if and only if it is a union of a collection of blocks of the quotient set A/R.
- If R and S are equivalence relations on A then the following statements are equivalent.
  - $R \subseteq S$.
  - Every S-saturated set is R-saturated.
  - Every block of the quotient set A/S is R-saturated.

20.3.5. Show that if A and B are finite sets, then $|B^A| = |B|^{|A|}$, justifying the notation.

20.3.6. Show that for any function f and any set A we have $f|_A = f \cap (A \times ran(f)$.

20.3.7. Show that an operation may have at most one (two-sided) identity element.

20.3.8. Let $(S, .)$ be a semigroup and for any two subsets of S as X and Y define

$$XY \overset{\text{def}}{=} \{x.y \ : \ x \in X \text{ and } y \in Y\}.$$

Now, for any three subsets X, Y and Z of S prove,

- $X(Y \cup Z) = XY \cup XZ$ and $(Y \cup Z)X = YX \cup ZX$.
- If $X \subseteq Y$ then $XZ \subseteq YZ$ and $ZX \subseteq ZY$.

20.3.9. Given finite sets Σ, explicitly introduce a one to one and onto map $\Sigma^* \xrightarrow{1-1} \mathbb{N}$ showing that Σ^* is a countable set.

20.3.10. Show that for any two languages K and L we have $K \backslash L = (L^R/K^R)^R$.

20.3.11. Study and analyze the properties of the quotient of languages with respect to union, intersection and concatenation of languages.

20.3.12. Given finite sets Σ and Γ, show that for any homomorphism $\sigma : \Sigma^* \bullet\!\!\longrightarrow \Gamma^*$ we have $\sigma(\epsilon) = \epsilon$.

20.3.13. Given finite sets Σ and Γ, show that any map $f : \Sigma \bullet\!\!\longrightarrow \Gamma$ may be extended to a homomorphism $\sigma_f : \Sigma^* \bullet\!\!\longrightarrow \Gamma^*$ in a unique way.

20.3.14. Show that a lattice can be defined as an algebraic structure with two binary operation.

20.3.15. Let R be a relation on X. Use what you learned in Section 20.1.7 to show that one may talk about the smallest equivalence relation $\bar{R}$ containing R, and the map $\overline{(-)}$ is actually a closure operator.

20.3.16. A simple graph $G(V, E)$ is said to be a *bipartite graph* if there is a partition of the vertex set as $V = A \cup B$ such that all edges appear between the parts A and B. Show that a simple graph is bipartite if it does not contain any odd cycle. Deduce that any simple tree is a bipartite graph.