THEORY OF COMPUTATION

Chapter7
Regular Languages
(Automata as Algebras)

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An algebra

Definition: An algebra is a pair $\mathcal{A} = (A, O)$ where A is a set and O is a set of operations on A.

- An $n-$ary operation τ on $\mathcal{A}$ is a function that takes n elements of $\mathcal{A}$ and returns a single element of $\mathcal{A}$; i.e.

$$\tau : \underbrace{A \times \ldots \times A}_{n} \to A,$$

- A $0-$ary operation (nullary operation) is simply an element of A, or a constant,
- A $1-$ary operation (unary operation) is simply a function from A to A,
- A $2-$ary operation (binary operation) is simply a function from $A \times A$ to A.

A pre-automaton

Definition: A pre-automaton

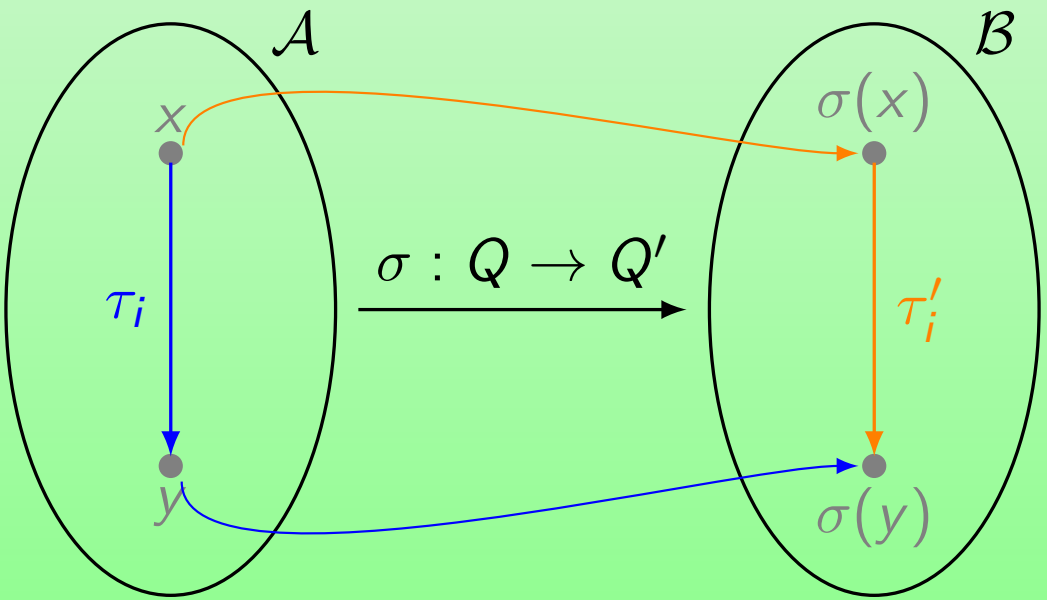
$$\mathcal{A} = \left(Q, q_0 \in Q, \{\tau_i : Q \to Q\}_{i \in \Sigma}\right)$$

with input symbols Σ of n elements, is an algebra with one nullary and n unary operations on a set of states A.

Homomorphisms

$$\mathcal{A} = \left(Q, q_0 \in Q, \{\tau_i : Q \to Q\}_{i \in \Sigma}\right)$$

$$\mathcal{B} = \left(Q', q_0{}' \in Q', \{\tau_i{}' : Q' \to Q'\}_{i \in \Sigma}\right)$$



$$\forall i \quad \tau_i' \circ \sigma = \sigma \circ \tau_i$$

$$\sigma(q_0) = q_0'$$

Isomorphisms

Definition: Two algebras

$$\mathcal{A} = \left(Q, q_0 \in Q, \{\tau_i : Q \to Q\}_{i \in \Sigma}\right)$$

and

$$\mathcal{B} = \left(Q', {q_0}' \in Q', \{{\tau_i}' : Q' \to Q'\}_{i \in \Sigma}\right)$$

are said to be isomorphic if there exists a bijective homomorphism $\sigma : \mathcal{A} \to \mathcal{B}$ such that its inverse, $\sigma^{-1} : \mathcal{B} \to \mathcal{A}$, is also a bijective homomorphism.

Subalgebras

- Let $\mathcal{A} = (A, O)$ be an algebra and let $\tau \in O$ be an $n-$ary operation of A. A subset B of A is $\tau-$ *closed* if $a_0, \ldots, a_{n-1} \in B$ imply $\tau(a_0, \ldots, a_{n-1}) \in B$.
- B is a *closed subset* of $\mathcal{A}$ if B is $\tau-$ closed for each operation τ of $\mathcal{A}$.
- A subalgebra of the algebra $\mathcal{A} = (A, O)$ is an algebra $\mathcal{B} = (B, O_B)$, where $B \subseteq A$ is a closed subset of A and

$$O_B = \{\tau|_{B^n} \ : \ \tau \in O \text{ is an } n-\text{ary operation}\}.$$

In other words, a subalgebra is a subset of an algebra which is closed under all its operations and carrying the induced operations.

- A reduced algebra is an algebra whose only subalgebras are the empty set and itself.

Reduced pre-automata

A reduced pre-automaton

$$\mathcal{A} = \left(Q, q_0 \in Q, \{\tau_i : Q \to Q\}_{i \in \Sigma}\right)$$

is a pre-automaton which is reduced as an algebra. Note that this is equivalent to the condition that each state of Q is reachable from the initial state q_0.

Equivalence relations

- An equivalence relation on X is a binary relation $\sim$ on X such that
  - $\forall x \in X \ (x \sim x)$ (reflexivity),
  - $\forall x, y \in X \ (x \sim y \rightarrow y \sim x)$ (simmetry),
  - $\forall x, y, z \in X \ (x \sim y \ \wedge \ y \sim z \rightarrow x \sim z)$ (transitivity).
- The equivalence class of an element x in X with respect to the equivalence relation $\sim$, denoted by $[x]_{\sim}$, is the set of all elements of X which are equivalent to x, i.e.

$$[x]_{\sim} = \{y \in X \mid x \sim y\}.$$

- Every two equivalence classes $[x]_\sim$ and $[y]_\sim$ are either equal or disjoint, so the set of equivalence classes is a partition of X.
- The set of all equivalence classes of X by $\sim$, denoted by $X/\sim$, is called the quotient set of X.
- For any partition $\mathcal{P}$ of a set X, the relation $\sim$ on X defined by

  $x \sim y$ if and only if x and y belong to the same element of $\mathcal{P}$,

  is an equivalence relation on X.

  Hence, talking about equivalence relations on a set is essentially the same as talking about partitions of a set: An equivalence relation determines a partition (in which the subsets are the equivalence classes), and a partition determines an equivalence relation (in which being equivalent means belonging to the same subset).

Equivalence coming from an onto-map

Any onto-map $\sigma : A \xrightarrow{onto} B$ gives rise to an equivalence relation on A according to which

$$a_1 \sim_\sigma a_2 \Leftrightarrow \sigma(a_1) = \sigma(a_1).$$

Hence, the collection of all $\sigma-$inverse images of the elements of B forms a partition of A.

Congruence relations

Definition: Let $\mathcal{A} = (A, O)$ be an algebra. A congruence relation is an equivalence relation on A which is compatible with the algebra operations. In the case of a pre-automaton where each operation is either nullary or unary, it means that for every x and y in A and each unary operation τ_i:

$$[x] = [y] \Leftrightarrow [\tau_i(x)] = [\tau_i(y)].$$

The quotient pre-automaton

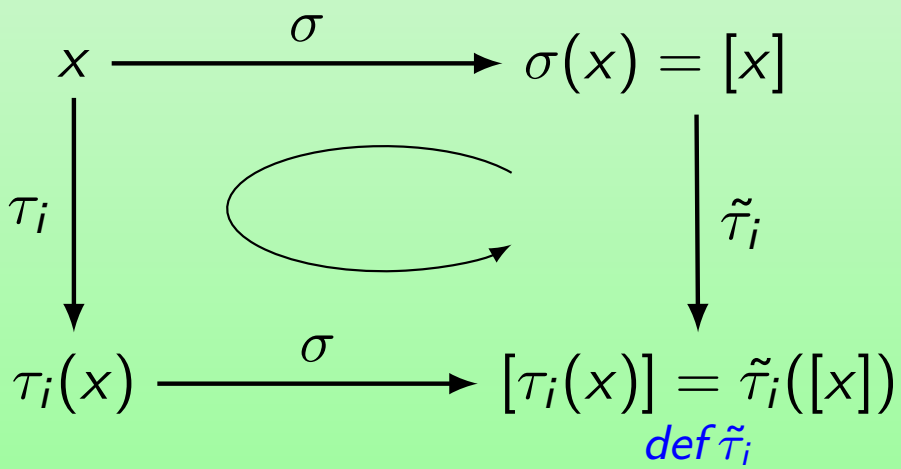
Definition: Let $\mathcal{A} = \left(Q, q_0 \in Q, \{\tau_i : Q \to Q\}_{i \in \Sigma}\right)$ be a pre-automaton and $\sim$ be a congruence relation on $\mathcal{A}$. The quotient pre-automaton of $\mathcal{A}$ by $\sim$ is

$$\mathcal{A}/\sim = \left(Q/\sim,\ [q_0],\ \{\tilde{\tau}_i : Q/\sim \to\ \ Q/\sim\}_{i \in I}\right)$$

where $\tilde{\tau}_i([x]) = [\tau_i(x)]$.

The canonical homomorphism

Consider the map $\sigma : \mathcal{A} \to \mathcal{A}/\sim$ defined as $\sigma(x) = [x]$; then σ is called the *natural* or the *canonical* map. Obviously σ is a homomorphism, or equivalently:



In other words

$$\sigma(\tau_i(x)) = [\tau_i(x)] = \tilde{\tau}_i(\sigma(x)).$$

and

$$\sigma(q_0) = [q_0].$$

Let

$$\mathcal{A} = \left(Q, q_0 \in Q, \{\tau_i : Q \to Q\}_{i \in \Sigma}\right)$$

and

$$\mathcal{B} = \left(Q', q_0{}' \in Q', \{\tau_i{}' : Q' \to Q'\}_{i \in \Sigma}\right)$$

be two pre-automata and $\sigma : \mathcal{A} \to \mathcal{B}$ be an onto homomorphism. Since σ maps $\mathcal{A}$ onto $\mathcal{B}$, the collection of all $\sigma-$inverse images of the elements of $\mathcal{B}$ forms a partiotion of $\mathcal{A}$. Let $\sim_\sigma$ be the equivalence relation corresponding to this partition. Since σ is a homomorphism, $\sim_\sigma$ is a congruence relation on $\mathcal{A}$:

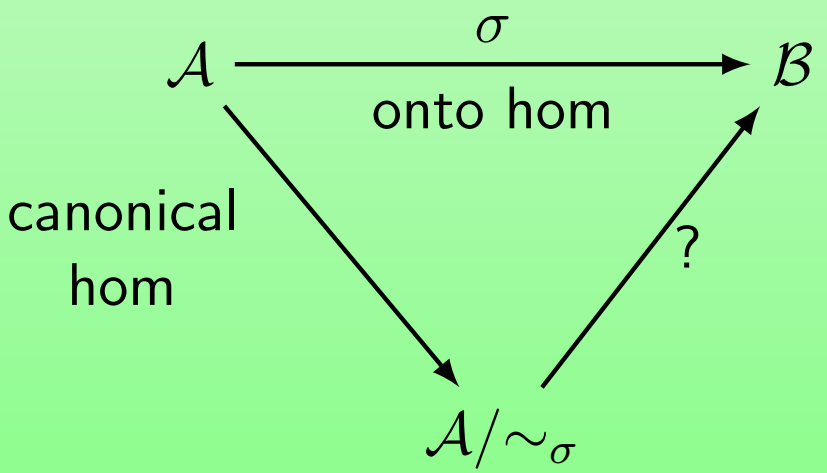
$$\begin{aligned}
[x]_{\sim_\sigma} = [y]_{\sim_\sigma} &\Leftrightarrow \sigma(x) = \sigma(y) \\
&\Leftrightarrow \tau_i{}'(\sigma(x)) = \tau_i{}'(\sigma(y)) \\
&\Leftrightarrow \sigma(\tau_i(x)) = \sigma(\tau_i(y)) \\
&\Leftrightarrow [\tau_i(x)]_{\sim_\sigma} = [\tau_i(y)]_{\sim_\sigma} \\
&\Leftrightarrow \tilde{\tau}_i([x]_{\sim_\sigma}) = \tilde{\tau}_i([x]_{\sim_\sigma}).
\end{aligned}$$

Hence, having an onto homomorphism $\sigma : \mathcal{A} \to \mathcal{B}$, one can make the quotient pre-automaton $\mathcal{A}$ by $\sim_\sigma$

$$\mathcal{A}/\sim_\sigma = \left(Q/\sim_\sigma,\ [q_0],\ \{\tilde{\tau}_i : Q/\sim_\sigma \ \to \ Q/\sim_\sigma\}_{i\in\Sigma}\right)$$

where $\tilde{\tau}_i([x]) = [\tau_i(x)]$.

Question:



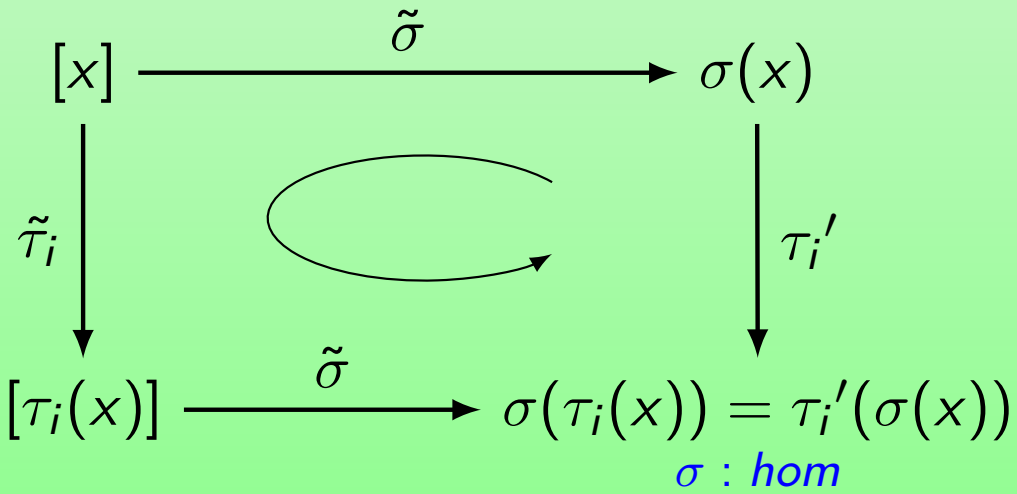
Consider the bijective map

$$\tilde{\sigma} : \mathcal{A}/\sim_\sigma \longrightarrow \mathcal{B}$$

$$[x] \mapsto \sigma(x)$$

$$[q_0] \mapsto \sigma(q_0) = {q_0}'$$

Then $\tilde{\sigma}$ is a homomorphism:



Consequently,

$$\mathcal{A}/\sim_\sigma \simeq \mathcal{B}.$$

Without loss of genelarity, assume that all pre-automata are defined on $\{0,1\}$.
So a pre-automaton is an algebra of type (0,1,1).

Consider the following fixed pre-automaton:

$$U_r = \left(\{0,1\}^*, \lambda \in \{0,1\}^*, \{r_i : \{0,1\}^* \to \{0,1\}^*\}_{i=0,1}\right)$$

where $r_i(w_1 \dots w_n) = w_1 \dots w_n i$.

Claim: For any reduced pre-automaton

$$\mathcal{B} = \left(P, p_0 \in P, \{\tau_i : P \to P\}_{i=0,1}\right)$$

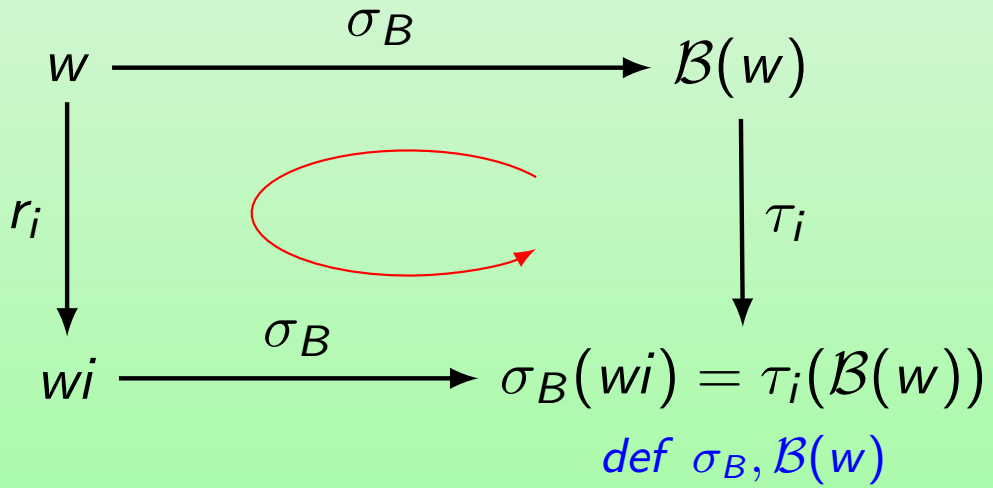
there exists an onto homomorphism $\sigma_B : U_r \xrightarrow{onto} \mathcal{B}$ such that $\sigma_B(w) = \mathcal{B}(w)$. In other words

$$\sigma_B(w_1 \dots w_n) = \tau_{w_n} \circ \dots \circ \tau_{w_2} \circ \tau_{w_1}(p_0).$$

Sketch of Proof.

$\mathcal{B}$: a reduced pre-automaton $\Rightarrow$ σ_B : an onto-map

and



Hence,

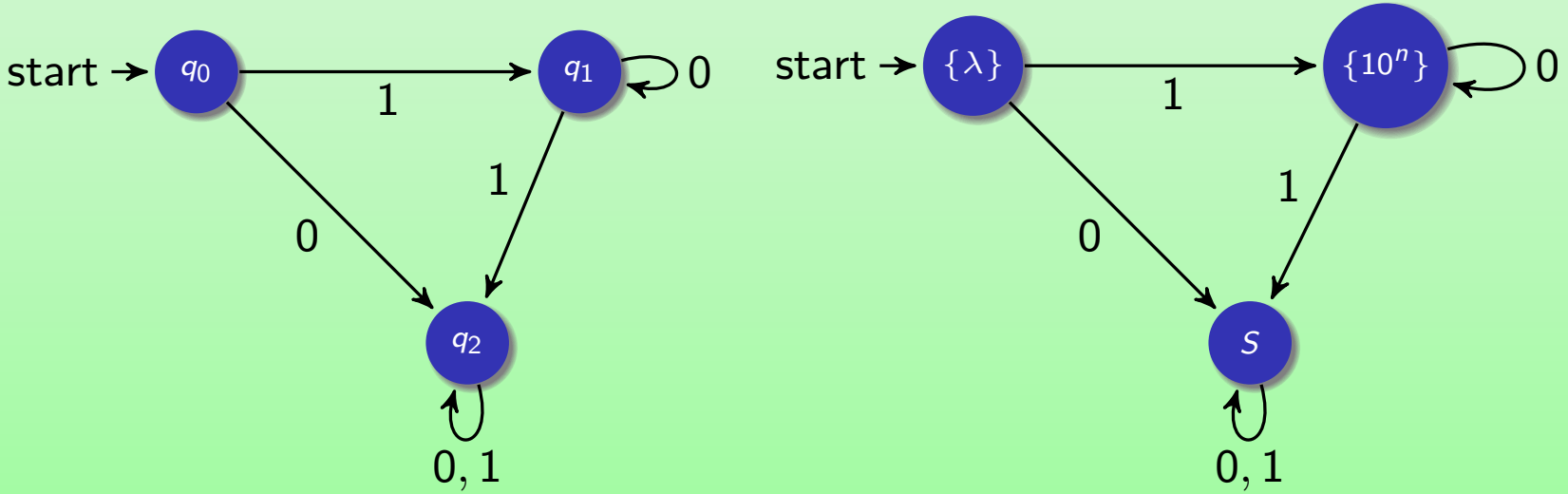
σ_B is a homomorphism.

Therefore,

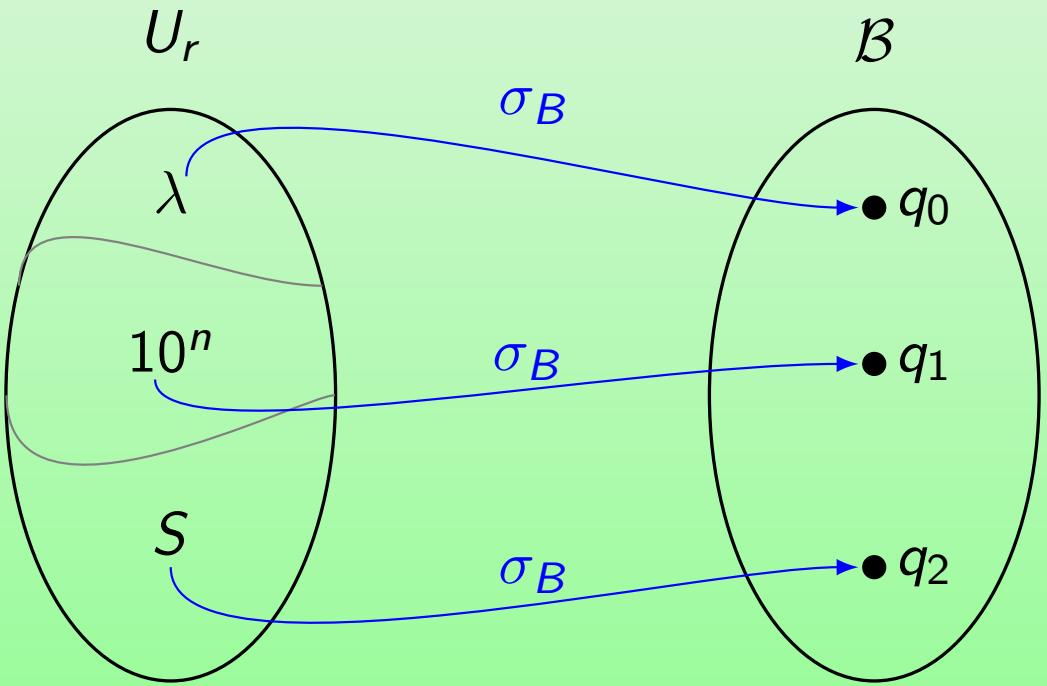
$$U_r/\sim_{\sigma_B} \simeq \mathcal{B}.$$

Example

The following two pre-automata are isomorphic:



where $S = \{0,1\}^* \setminus (\{\lambda\} \cup \{10^n \mid n \in \mathbb{N}\})$.


U_r
$\mathcal{B}$
σ_B
λ
q_0
10^n
σ_B
q_1
S
σ_B
q_2

The minimal automaton

Let $L \subseteq \{0,1\}^*$ be a language. We are trying to construct a minimal pre-automaton $\mathcal{M}_L$ to accept L. We will be needing some ingredients as follows:

- Definition: For a language $L \subseteq \{0,1\}^*$ and any $x \in \{0,1\}^*$, define:

$$x\backslash L = \{u \in \{0,1\}^* \mid xu \in L\}.$$

From this definition, it follows that for every $x, y \in \{0,1\}^*$:

$$\begin{aligned} x\backslash(y\backslash L) &= \{u \in \{0,1\}^* \mid xu \in y\backslash L\} \\ &= \{u \in \{0,1\}^* \mid yxu \in L\} \\ &= (yx)\backslash L. \end{aligned}$$

- Definition: For a language $L \subseteq \{0,1\}^*$, define a relation $\sim$ on $\{0,1\}^*$ as follows: for $x, y \in \{0,1\}^*$,

$$x \sim y \quad \Leftrightarrow \quad x\backslash L = y\backslash L$$
$$\Leftrightarrow \quad \{u \in \{0,1\}^* \mid xu \in L\} = \{u \in \{0,1\}^* \mid yu \in L\}.$$

In other words, $x \sim y$ if and only if for any $u \in \{0,1\}^*$, xu and yu are either both in L or both are not in L.

- Note: The relation $\sim$ is indeed an equivalence relation; It is easy to see that the relation $\sim$ is reflexive, symmetric, and transitive, because the equality relation has these properties.

Now we are ready to construct the minimal pre-automaton accepting L.

Claim: For a language $L \subseteq \{0,1\}^*$, $\mathcal{M}_L = (Q\ ,\ q_0\ ,\ \tau_0\ ,\ \tau_1)$ is the minimal pre-automaton w.r.t. right concatenation accepting L, where

1. $Q = \{x\backslash L \mid x \in \{0,1\}^*\}$,
2. $q_0 = \lambda\backslash L$,
3. $\tau_i(x\backslash L) = i\backslash(x\backslash L) = (xi)\backslash L, \qquad i = 0, 1.$

Example

$L = \{1\,0^n \mid n \in \mathbb{N}\}$

$\lambda \backslash L = L$

$1 \backslash L = \{0^n \mid n \in \mathbb{N}\} = Z$

$1\,0^j \backslash L = Z \qquad j \in \mathbb{N}$

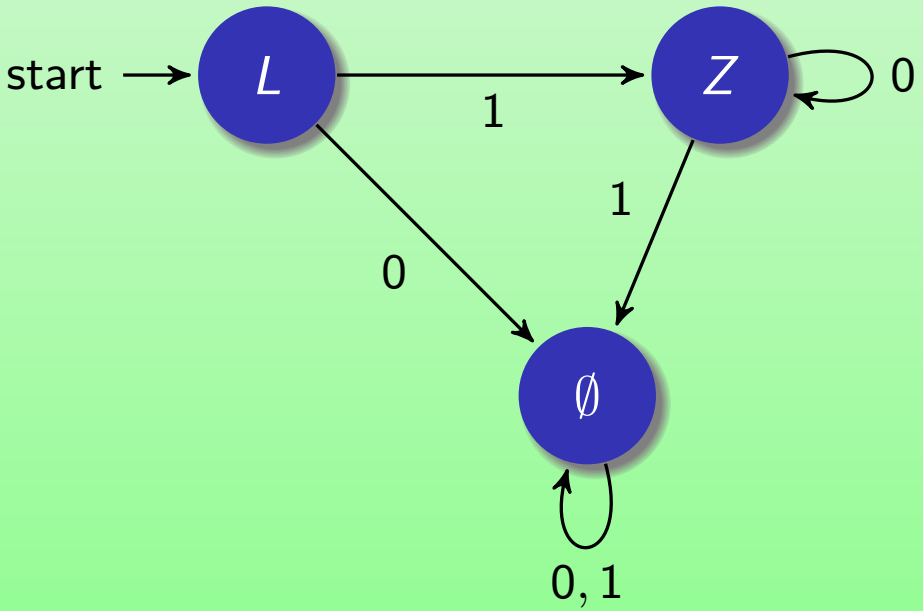
$0 \backslash L = \emptyset$

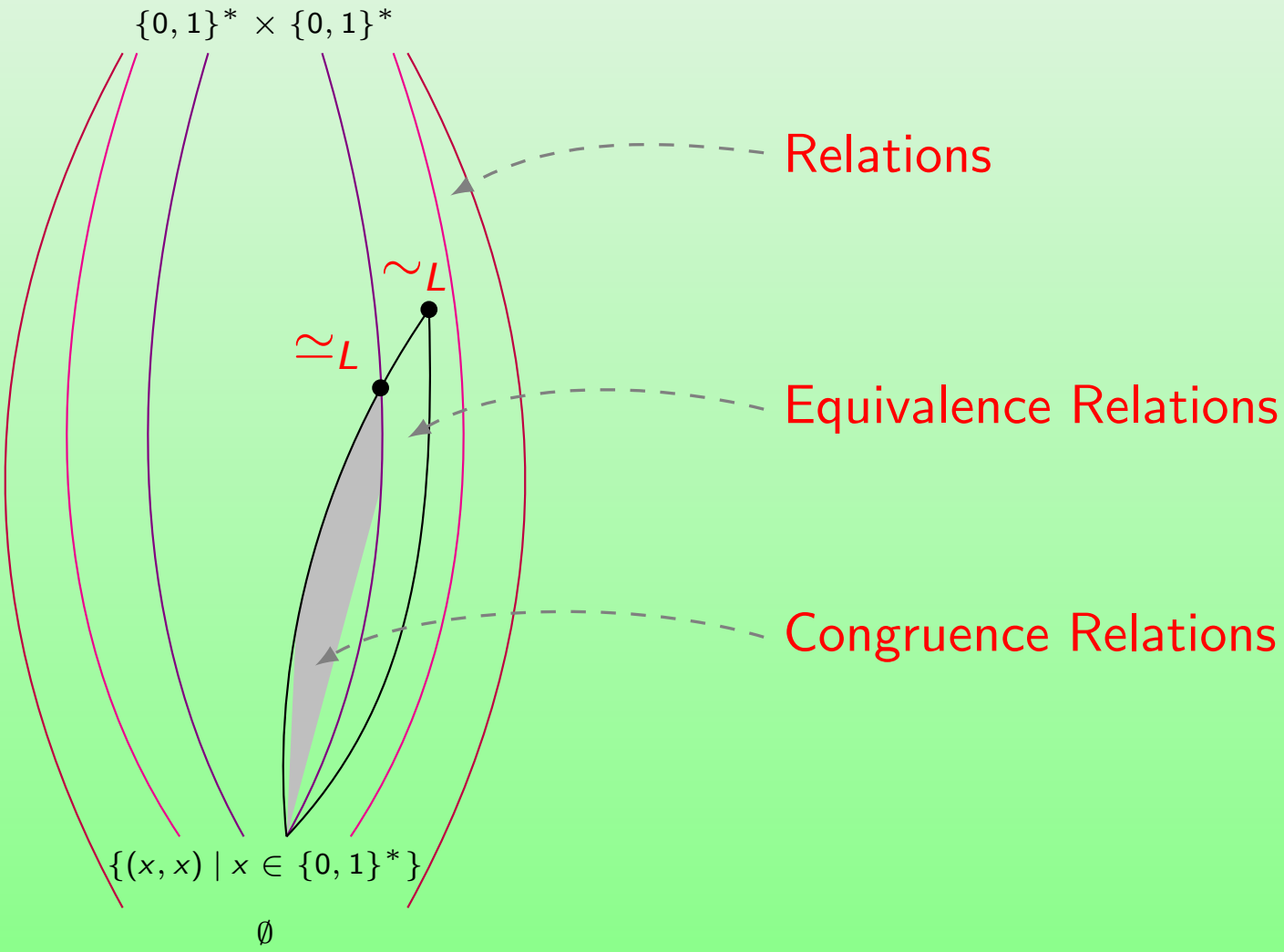
$0u \backslash L = \emptyset \qquad u \in \{0,1\}^*$

$11u \backslash L = \emptyset \qquad u \in \{0,1\}^*$

$101u \backslash L = \emptyset \qquad u \in \{0,1\}^*$

$\vdots$




$\{0,1\}^* \times \{0,1\}^*$
$\sim_L$
$\sim_L$
$\{(x,x) \mid x \in \{0,1\}^*\}$
$\emptyset$
Relations
Equivalence Relations
Congruence Relations

- A language $L \subseteq \{0,1\}^*$ determines an equivalence relation $\sim_L$ on $\{0,1\}^*$ in a usual way, i.e. for $x, y \in \{0,1\}^*$

$$x \sim_L y \;\Leftrightarrow\; x, y \in L \;\vee\; x, y \in L^c.$$

- Now we define a congruence relation $\simeq_L$ on $\{0,1\}^*$ as follows: for $x, y \in \{0,1\}^*$,

$$x \simeq_L y \;\Leftrightarrow\; x\backslash L = y\backslash L.$$

- Claim: $\simeq_L$ is the greatest congruence relation less than $\sim_L$.

Claim1: The relation $\simeq_L$ is a congruence relation on U_r.

Proof. It is easy to see that $\simeq_L$ is an equivalence relation on $\{0,1\}^*$, so it suffices to show that $\simeq_L$ is compatible with r_i, for i=0,1:

$$
\begin{aligned}
x \simeq_L y &\Leftrightarrow x\backslash L = y\backslash L \\
&\Leftrightarrow \forall u \in \{0,1\}^* \ (xu, yu \in L \ \vee \ xu, yu \notin L) \\
&\Leftrightarrow \forall u \in \{0,1\}^* \ \forall i \in \{0,1\} \ (xiu, yiu \in L \ \vee \ xiu, yiu \notin L) \\
&\Leftrightarrow xi\backslash L = yi\backslash L \\
&\Leftrightarrow xi \simeq_L yi \\
&\Leftrightarrow r_i(x) \simeq_L r_i(y).
\end{aligned}
$$

Claim2: $\simeq_L \subseteq \sim_L$.

Proof.

$$
\begin{aligned}
x \simeq_L y &\Rightarrow x\backslash L = y\backslash L \\
&\Rightarrow \forall u \in \{0,1\}^* \ (xu, yu \in L \ \vee \ xu, yu \notin L) \\
&\Rightarrow \text{for } \lambda \in \{0,1\}^* \ (x\lambda, y\lambda \in L \ \vee \ x\lambda, y\lambda \notin L) \\
&\Rightarrow x, y \in L \ \vee \ x, y \notin L \\
&\Rightarrow x, y \in L \ \vee \ x, y \in L^c \\
&\Rightarrow x \sim_L y.
\end{aligned}
$$

...

Claim3: For any congruence relation ρ on $\{0,1\}^*$,

$$\rho \subseteq \sim_L \Longrightarrow \rho \subseteq \simeq_L .$$

Proof.

$$\begin{aligned}
(x,y) \in \rho &\Rightarrow \forall i \in \{0,1\}\ (r_i(x), r_i(y)) \in \rho \\
&\Rightarrow \forall i \in \{0,1\}\ (xi, yi) \in \rho \\
&\Rightarrow \forall u \in \{0,1\}^*\ (xu, yu) \in \rho \\
&\Rightarrow \forall u \in \{0,1\}^*\ xu \sim_L yu \\
&\Rightarrow \forall u \in \{0,1\}^*\ (xu, yu \in L \ \vee \ xu, yu \in L^c) \\
&\Rightarrow \forall u \in \{0,1\}^*\ (xu, yu \in L \ \vee \ xu, yu \notin L) \\
&\Rightarrow x\backslash L = y\backslash L \\
&\Rightarrow x \simeq_L y.
\end{aligned}$$

Now we can form the quotient pre-automaton $U_r/\simeq_L$ and show that

$$U_r/\simeq_L \ \simeq \ \mathcal{M}_L.$$

Therefore, it follows from the construction of $U_r/\simeq_L$ that

$\mathcal{M}_L$ is the minimal pre-automaton w.r.t. right concatenation accepting L.

In order to show this isomorphism, consider the map $\sigma : U_r \to \mathcal{M}$ by $\sigma([x]_{\simeq_L}) = \mathcal{M}(x) = x\backslash L$ for any $x \in \{0,1\}^*$.

- σ is a well-defined map:

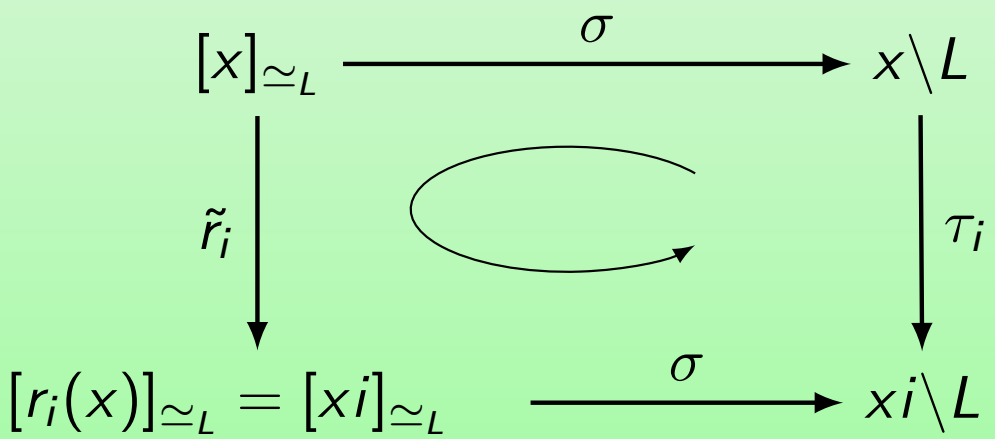
  1. For any $x = x_1 \dots x_n$:

$$\begin{aligned}
\mathcal{M}(x) &= \mathcal{M}(x_1 \dots x_n) \\
&= \tau_{x_n} \circ \dots \circ \tau_{x_2} \circ \tau_{x_1}(\lambda \backslash L) \\
&= \tau_{x_n} \circ \dots \circ \tau_{x_2}(x_1 \backslash L) \\
&= \tau_{x_n} \circ \dots \circ \tau_{x_3}(x_1 x_2 \backslash L) \\
&\vdots \\
&= x_1 \dots x_n \backslash L \\
&= x \backslash L.
\end{aligned}$$

  2. For any $x, y \in \{0,1\}^*$:

$$[x]_{\simeq_L} = [y]_{\simeq_L} \Leftrightarrow x \simeq_L y \Leftrightarrow x \backslash L = y \backslash L.$$

- σ is a homomorphism:



$$\tau_i \circ \sigma = \sigma \circ \tilde{r}_i.$$

- Obviously, σ is a bijective map.

In a similar way, we can treat L w.r.t. left concatenation as follows:

Definition: For a language $L \subseteq \{0,1\}^*$ and any $x \in \{0,1\}^*$, define:

$$L\backslash x = \{u \in \{0,1\}^* \mid ux \in L\}.$$

It follows from the definition that for every $x, y \in \{0,1\}^*$:

$$\begin{aligned} (L\backslash x)\backslash y &= \{u \in \{0,1\}^* \mid uy \in L\backslash x\} \\ &= \{u \in \{0,1\}^* \mid uyx \in L\} \\ &= L\backslash(yx). \end{aligned}$$

Hence, for a language $L \subseteq \{0,1\}^*$ the minimal pre-automaton w.r.t. left concatenation accepting L is $\mathcal{N}_L = (Q\ ,\ q_0\ ,\ \tau_0\ ,\ \tau_1)$, where

1. $Q_L = \{L\backslash x \mid x \in \{0,1\}^*\}$,
2. $q_0 = L\backslash\lambda$,
3. $\tau_i(L\backslash x) = (L\backslash x)\backslash i = L\backslash(ix)$. $\qquad i = 0, 1$.

Example

$L = \{1\,0^n \mid n \in \mathbb{N}\}$

$$L\backslash\lambda = L$$
$$L\backslash 0 = L$$
$$L\backslash 0^j = L \qquad j \in \mathbb{N}$$
$$L\backslash 1 = \{\lambda\}$$
$$L\backslash 1\,0^j = \{\lambda\} \qquad j \in \mathbb{N}$$
$$L\backslash u = \emptyset$$
$$u \neq \lambda,\ u \neq 0^j, 1\,0^j \qquad j \in \mathbb{N}$$