Take-Home Exam
Discrete Geometric Optimization

Date: 98.11.8

Time: Start (9am 98.11.8) – Stop (9am 98.11.10)

***You can consult anyone or anything except your classmates.**

!Unreadable documents will not go through the grading procedure.

1. Consider the Petersen graph, P, and answer the following question (you are free for your choice if there is any degree of freedom!):

    a) Compute the combinatorial Laplacian and the Laplacian of the natural random walk of P. (Explain the differences if any.)

    b) Do the spectral clustering for $k = 2$ and $k = 3$ parts when all edge weights are equal to one. Does the first eigenvector provide any information? What happens if we disregard the first eigenvector? (Report the result and add your conclusions.)

    c) Do the max and mean isoperimetric clustering for $k = 3$ parts when all edge weights are equal to one. Is there any minimizer which is a partition? (Report the result and add your conclusions comparing with results of Part b.)

    d) Compute the strong nodal domains of the third eigenfunction (counting equal eigenvalues!) and compare to Part c. Also, compute the Dirichlet eigenvalues of these strong nodal domains.

    e) Choose one of the edges arbitrarily and perturb its weight to $1+\epsilon$. Answer parts c and d for this perturbed graph and compare your results when ϵ tends to zero.

    f) Choose one of the edges arbitrarily and perturb its weight to $1-\epsilon$. Answer parts c and d for this perturbed graph and compare your results when ϵ tends to one.

    g) Compute the zeta-function of P. Is P a Ramanujan graph? (Why?) Is your answer related to the symmetry of the graph? What can you say about the graph $P - e$ for an arbitrary edge e?

    h) Choose four clustering measures from Parts b and c in the Wasserstein space. Compute their entropy and compare the results. Compute the largest distance between these four measures and justify your result comparing these two farthest measures.

i) Find the barycenter of the four measures of Part h in the Wasserstein space and explain how it is related to the graph itself.

j) Develop a clustering algorithm based on the Wasserstein distance by providing a suitable cost function. Discuss the complexity of the optimization problem and provide an algorithm to solve it (possibly in an approximation scenario).

k) Using your results of the previous parts, provide your best metric embedding of the Petersen graph into $\mathbb{R}$ (i.e. the best distortion you can get).

2. Consider a random bipartite graph on parts A and B with $|A| = |B|$ for which each edge between the parts A and B are added with a constant probability $0 < p < 1$.

   a) Apply the cavity method to partition such a random graph into $k = 2$ and $k = 3$ parts (you are asked to write down the cavity recursive equations).

   b) Linearize your dynamics and justify the fact that the linearization is related to the nonbacktracking matrix M. Analyze the cavity dynamics in this scenario and justify your answers using intuition.

   c) Does the spectrum of the nonbacktracking matrix provide any information in this case?

   d) Does the spectrum of $\frac{1}{2}(M + M^*)$ provide any information in this case? What about the symmetrization we explained in the class to obtain a Cheeger-type inequality?